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MATHEMATICAL QUESTIONS

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

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W. J. C. MILLER, B.A.,

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Mathematical Papers, &c.

Solved Questions.

CORRIGENDA IN VOL. XXXI.

Page 24, line 7, *for* $27(x^2+y^2)^2$ *read* $27c^2(x^2+y^2)^2$.

Page 24, line 8, *for* $(x^2+y^2-4c^2)^2$ *read* $(x^2+y^2-4c^2)^3$.

Page 24, line 2 from bottom, *for* $\sin\theta$ *read* $\tan\theta$.

Page 39, line 2, *for* 8 *read* δ, and *for* 28 *read* 2δ.

Page 50, line 6 from bottom, *for* a *read* a_1.

Page 51, line 8 from bottom, *for* they *read* then.

Page 51, line 15 from bottom, *for* $(s-1)^p$ *read* $(s-1\,!)^p$;

Page 56, last line, *for* -2730 *read* $= 2730$.

Page 111, line 14, *for* $+q+xr$ *read* $+qx+r$.

MATHEMATICS

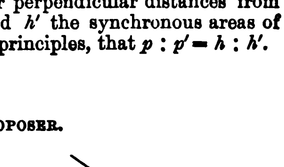
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5690. (By Professor TOWNSEND, F.R.S.)—Two planets, moving in different planes and describing any orbits round the Sun as common centre of attraction, being supposed to appear stationary in the heavens as observed from each other: if p and p' be their perpendicular distances from the line of nodes of their orbits, and h and h' the synchronous areas of their orbital motions, shew, on elementary principles, that $p : p' = h : h'$.

Solution by the PROPOSER.



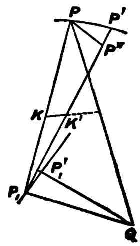
If P and P′ be the positions of the two planets at the instant in question, Q and Q′ their positions at the consecutive instant, R and R′ the feet of the perpendiculars from them upon the line of nodes SN of their orbits, and S the position of the sun; then, since by hypothesis PP′ and QQ′ are parallel to each other, therefore PQ and P′Q′ intersect at some point N in the line of nodes SN of the two orbits, and PN : P′N = PQ : P Q′; hence area PSN : area P′SN = area PSQ : area P′SQ′; but area PSN : area P′SN = PR : P′R′ = $p : p'$, and area PSQ : area P′SQ′ = $h : h'$; and therefore, &c.

5766. (By Professor MINCHIN, M.A.)—The vector to any point on a plane curve being ρ, and the successive s-fluxes of ρ being denoted by ρ', ρ'', ρ''', &c., give a simple construction for the direction of the vector ρ'''.

Solution by J. J. Walker, M.A.

It is known that ρ'' is represented by a vector parallel to the normal—say, at P—and equal to the reciprocal of the radius (R) of curvature. Hence $\dot{\rho}''$ will be represented by a vector parallel to the element of the curve which is the locus of the end of a portion of ρ'' equal to that reciprocal; which element will be perpendicular to the line joining the point P with the centre of curvature—say Q—of the evolute at the point corresponding to P. As regards magnitude, it appears without much difficulty, from the geometry of the figure, that $\rho''' = \dot{\rho}'' : \dot{s} = \dfrac{PQ}{R^3}$.

For, let P′ be consecutive to P, and let P_1, P_1' be the corresponding points on evolute; let $P_1K = \dfrac{1}{PP'}$, and taking $P'P'' = P_1P_1'$, let $P_1K' = \dfrac{1}{P'P_1'}$. Then KK′ is parallel to $\dot{\rho}''$; and, in magnitude,



$$\dot{\rho}'' = \frac{KK'}{\delta t} = \frac{PP''}{R^2\delta t}.$$

Now $$PP''^2 = PP'^2 + P'P''^2,$$

or $$P_1P_1'^2 = (PP_1^2 + QP_1^2)(\delta\phi)^2 = (PQ\,.\,\delta\phi)^2,$$

therefore $$\dot{\rho}'' = \frac{PQ}{R^2}\dot{\phi};$$

and finally $$\rho''' = \dot{\rho}'' \div \dot{s} = \frac{PQ}{R^2}\frac{\dot{\phi}}{\dot{s}} = \frac{PQ}{R^3}.$$

As regards direction,

$$\tan K' = \tan P'' = PP' : P'P'' = PP' : P_1P_1' = PP_1 : QP_1 = \tan Q.$$

Hence KK′ is perpendicular to PQ. [See Hamilton's *Elements*, p. 533, and Tait's *Quaternions*, 1st ed., p. 198.]

5527. (By W. S. B. Woolhouse, F.R.A.S., &c.)—A given triangle may be orthogonally projected from an equilateral triangle; or it may be orthogonally projected into an equilateral triangle. Determine, by an easy geometrical construction, the magnitudes of these equilateral triangles.

I. *Solution by the* Proposer.

Let an equilateral triangle PQR, having the equal sides E, be orthogonally projected vertically into the triangle ABC, having the sides a, b, c in a horizontal plane. Then, considering the neutralization of the result of the changes of altitude in passing from Q to R, from R to P, and thence from P to the starting point Q, there arises the relation

$$(E^2-a^2)^{\frac{1}{2}} + (E^2-b^2)^{\frac{1}{2}} + (E^2-c^2)^{\frac{1}{2}} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

This equation can subsist only when the three terms of the expression are either all real or all imaginary. In the latter case, the imaginary char-

acter is removed by multiplying by $(-1)^{\frac{1}{2}}$, that is, by writing the equation thus: $(a^2-E^2)^{\frac{1}{2}}+(b^2-E^2)^{\frac{1}{2}}+(c^2-E^2)^{\frac{1}{2}}=0$ .... (2).

But the form (1) implies that ABC is projected from PQR; and (2) implies that ABC is projected into PQR. Hence, when cleared of radicals, the algebraic relation amongst the sides a, b, c, and E will be identically the same whether the triangle ABC be projected into or projected from the triangle PQR.

If Δ denote the area of the triangle ABC, then

$$16\Delta^2 = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \quad \text{.................. (3).}$$

To free (1) or (2) from surds, conceive the three terms of the expression to be the sides of a supposed triangle; then the area of such triangle is obviously zero. Hence, substituting in the formula for $16\Delta^2$, we get

$$0 = 2(E^2-a^2)(E^2-b^2)+2(E^2-a^2)(E^2-c^2)+2(E^2-b^2)(E^2-c^2) \\ -(E^2-a^2)^2-(E^2-b^2)^2-(E^2-c^2)^2;$$

which, arranged according to powers of E, is, by (3),

$$0 = 3E^4-2(a^2+b^2+c^2)E^2+16\Delta^2.$$

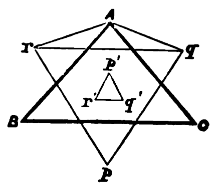
Let E_1^2, E_2^2 denote the two roots of this quadratic; then

$$E_1^2+E_2^2=\frac{2(a^2+b^2+c^2)}{3} \quad \text{and} \quad E_1^2E_2^2=\frac{16\Delta^2}{3};$$

therefore
$$(E_1\pm E_2)^2=\frac{2(a^2+b^2+c^2)}{3}\pm\frac{8\Delta}{3^{\frac{1}{2}}} \quad \text{.................. (4).}$$

To effect a geometrical construction of the problem, this result admits of being reduced as follows:—

$$(E_1\pm E_2)^2=\frac{4(b^2+c^2)}{3}-\frac{2(b^2+c^2-a^2)}{3}\pm\frac{4bc}{3^{\frac{1}{2}}}\sin A$$
$$=\tfrac{4}{3}\left\{b^2+c^2-bc(\cos A\mp 3^{\frac{1}{2}}\sin A)\right\}$$
$$=\tfrac{4}{3}\left\{b^2+c^2-2bc\cos(A\pm 60^\circ)\right\} \quad \text{........................ (5).}$$



Now, if p, q, r be the centres of equilateral triangles described externally upon the respective sides BC, CA, AB; and p', q', r' the centres of another set of equilateral triangles described internally, we shall have $Aq^2=\frac{b^2}{3}$, $Ar^2=\frac{c^2}{3}$, and the angle $qAr = A+60^\circ$; also $Aq'^2=\frac{b^2}{3}$, $Ar'^2=\frac{c^2}{3}$, and the angle $q'Ar'=A-60^\circ$.

Hence
$$\left.\begin{matrix} qr^2 \\ q'r'^2 \end{matrix}\right\}=\frac{b^2}{3}+\frac{c^2}{3}-\frac{2bc}{3}\cos(A\pm 60^\circ);$$

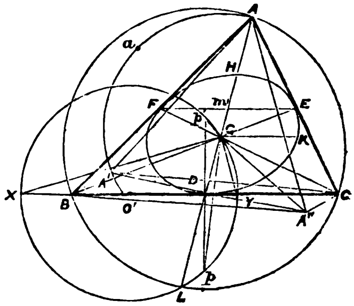
and therefore, by (5), $(E_1\pm E_2)^2=\left\{\begin{matrix}4qr^2\\4q'r'^2\end{matrix}\right.$, which gives

$$E_1\pm E_2=\left\{\begin{matrix}2qr\\2q'r'\end{matrix}\right. \quad \text{and} \quad \left.\begin{matrix}E_1\\E_2\end{matrix}\right\}=qr\pm q'r' \quad \text{............... (6).}$$

The symmetry of the expression (4), with respect to a, b, c, shows that the subsidiary triangles pqr, $p'q'r'$ are each of them equilateral; and, by (6), the magnitudes of the sides, E_1, E_2, of the required equilateral triangles are respectively the sum and difference of the sides of these subsidiary equilateral triangles.

II. *Solution by the* PROPOSER.

Let ABC be the given triangle; D, E, F the bisections of the sides BC, CA, AB, respectively; and DEF an inscribed ellipse touching the sides at those points. Then, if the triangle be placed in such a position that the ellipse is projected into a circle, the points of contact will still be at the bisections of the sides, and the projected triangle must necessarily be equilateral and have the diameter of the inscribed circle equal to the minor principal diameter of the ellipse.



The problem is thus resolved into an investigation of a geometrical construction of the magnitudes of the principal diameters of the ellipse inscribed in the given triangle.

As the chord FE bisects AD, it also bisects the semidiameter GH in m; also, since the chord FE is parallel to the tangent BC, it is conjugate to the diameter HD. Let GK, drawn parallel to BC, be the conjugate semidiameter; then $\quad \mathrm{H}m \,.\, m\mathrm{D} : m\mathrm{E}^2 :: \mathrm{GH}^2 : \mathrm{GK}^2.$

But $\mathrm{H}m = \frac{1}{2}\mathrm{GH}$, $m\mathrm{D} = \frac{3}{2}\mathrm{GH}$; therefore $\mathrm{H}m \,.\, m\mathrm{D} = \frac{3}{4}\mathrm{GH}^2$, and thence $m\mathrm{E}^2 = \frac{3}{4}\mathrm{GK}^2$, that is, $\mathrm{GK}^2 = \frac{4}{3}m\mathrm{E}^2 = \frac{1}{3}\mathrm{DC}^2$.

Let p, p' be the centres of equilateral triangles described below and upon BC; then $\mathrm{D}p^2 = \mathrm{D}p'^2 = \frac{1}{3}\mathrm{DC}^2$. Therefore $\mathrm{GK} = \mathrm{D}p = \mathrm{D}p'$.

Now, by Conics, if X, Y be two points in the tangent BC such that $\mathrm{XD} \,.\, \mathrm{DY} = \mathrm{GK}^2$, lines drawn from the centre G through these points will determine conjugate diameters of the ellipse; and they will coincide with the principal axes when the angle at G is a right angle, that is, when the circle (O′), described on XY as diameter, passes through G. Since $\mathrm{XD} \,.\, \mathrm{DY} = \mathrm{GK}^2 = \mathrm{D}p^2 = \mathrm{D}p'^2$, this circle passes also through the points p, p'.

Again, if AD be produced to meet the circumscribing circle in L, $\mathrm{D}p^2 = \frac{1}{3}\mathrm{DC}^2 = \frac{1}{3}\mathrm{AD} \,.\, \mathrm{DL} = \mathrm{GD} \,.\, \mathrm{DL}$; therefore the circle (O′) passes also through the point L. Any three of the four known points G, p, p', L will therefore serve to construct the circle (O′) geometrically; or the axes GX, GY may be drawn to bisect the interior and exterior angles formed by the known lines Gp, Gp'.

To determine the magnitudes of the principal semi-diameters, α, β, we shall have

$$\begin{aligned}
\alpha^2 + \beta^2 &= \mathrm{GD}^2 + \mathrm{GK}^2 = \mathrm{GD}^2 + \mathrm{D}p^2 = \tfrac{1}{2}(\mathrm{G}p^2 + \mathrm{G}p'^2) \\
2\alpha\beta &= 2\mathrm{GD} \,.\, \mathrm{GK} \sin \mathrm{DGK} \quad = 2\mathrm{GD} \,.\, \mathrm{D}p \cos \mathrm{GD}p' \\
&= \mathrm{GD}^2 + \mathrm{D}p'^2 - \mathrm{G}p'^2 \quad = \tfrac{1}{2}(\mathrm{G}p^2 + \mathrm{G}p'^2) - \mathrm{G}p'^2 \\
&\quad = \tfrac{1}{2}(\mathrm{G}p^2 - \mathrm{G}p'^2).
\end{aligned}$$

The sum and difference of these give $(\alpha+\beta)^2 = \mathrm{G}p^2$, $(\alpha-\beta)^2 = \mathrm{G}p'^2$. Hence $2\alpha = \mathrm{G}p + \mathrm{G}p'$, $2\beta = \mathrm{G}p - \mathrm{G}p'$.

The positions and magnitudes of the principal diameters are thus found by an easy construction.

The major diameter (2α) is evidently that of the circle inscribed in the equilateral triangle, from which the given triangle may be projected; and the minor diameter (2β) is that of the circle inscribed in the equilateral

triangle, into which the given triangle can be projected. The magnitudes of the required equilateral triangles are thus completely determined.

The geometrical construction, however, may be effected by a yet more simple and direct method. With vertex A and centre G, suppose an equilateral triangle AA′A″ to be drawn, and let straight lines connect the point A′ with B and C. Then the triangles $p'Gp$ and BA′C have the lines GD, A′D respectively bisecting the opposite sides in angles GDp', A′DB, which are equal; moreover, the bisectors have to each other evidently the same ratio as that of the diameter of the inscribed circle to the side of an equilateral triangle, and the bisected sides also have to each other the same ratio. The triangles $p'Gp$, BA′C are therefore similar in all respects, and the two sides Gp, Gp' have to the corresponding sides A′C, A′B the fixed ratio just stated. Hence, as it has been shown that the diameters of the circles inscribed in the required equilateral triangles are equal to the sum and difference of the lines Gp, Gp', it follows that the values of the sides of those triangles are respectively the sum and difference of the lines A′C, A′B; the sum of the said lines being the side of the equilateral triangle from which the given triangle may be orthogonally projected, and the difference of the same lines the side of the equilateral triangle into which the given triangle may be orthogonally projected.

The position of the point A′ is easily determinable by the compasses thus: With centre G and radius GA, describe the circle AaA'..., and on the circumference lay off the same radius from A to a, and from a to A′. On the other side of AG the point A″ is similarly found. It is obviously immaterial whether the point A′ or A″ be used. Also the same results should necessarily follow when either of the other angular points B, C is first used instead of A. This curious theorem is of course involved in the method of construction, and all the points may be simultaneously deduced by conceiving the given triangle to revolve about its centre G through angles equal to one-third and two-thirds of a circumference, and so to occupy the positions A′B′C′, A″B″C″.

The axes GX, GY are respectively parallel to the lines which bisect the exterior and interior angles (A′) of the triangle BA′C.

NOTE. 1.—The axes GX, GY are the mechanical principal axes of the given triangle ABC considered as a thin lamina. An investigation of their position has always seemed to me to be wanting in works on Dynamics.

NOTE 2.—*Theorem.*—The triangles AB″C′, A′BC″, A″B′C, AB′C″, A′B″C, A″BC′ are all equilateral.

NOTE 3.—*Theorem.*—By proceeding respectively from B and C as vertices, instead of A, two other circles analogous to (O′) are obtained. The centres of the three circles (O′) are in a right line; and they all intersect the circumference of the circumscribing circle of the given triangle in one common point, from the perpendiculars upon the sides of the triangle, produced, if necessary, are respectively

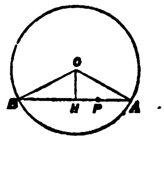
$$\frac{2\Delta a\,(a^2-b^2)\,(a^2-c^2)}{a^2(a^2-b^2)(a^2-c^2)+b^2(b^2-a^2)(b^2-c^2)+c^2(c^2-a^2)(c^2-b^2)},$$

$$\frac{2\Delta b\,(b^2-a^2)\,(b^2-c^2)}{a^2(a^2-b^2)(a^2-c^2)+b^2(b^2-a^2)(b^2-c^2)+c^2(c^2-a^2)(c^2-b^2)},$$

$$\frac{2\Delta c\,(c^2-a^2)\,(c^2-b^2)}{a^2(a^2-b^2)(a^2-c^2)+b^2(b^2-a^2)(b^2-c^2)+c^2(c^2-a^2)(c^2-b^2)}.$$

5721. (By Elizabeth Blackwood.)—A line is drawn at random through a random point within a circle, and another line is drawn at random through the centre; find the chance that these lines will intersect within the circle.

Solution by E. B. Seitz.



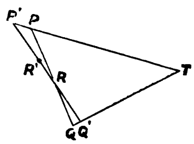
Let O be the centre of the circle, P the random point, and AB a chord through P. Draw OH perpendicular to AB. Now it is evidently necessary to consider only those chords through P which have a given direction. If the random line through O falls in the angle AOB, the two lines will intersect within the circle. Let OA $= r$, AP $= x$, $\angle$AOH $= \theta$. Then AB $= 2r\sin\theta$; the limits of θ are 0 and $\frac{1}{2}\pi$; and those of x are 0 and $2r\sin\theta$. Hence the required chance is

$$p = \frac{\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta} 4\theta r\sin\theta\, d\theta\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta} 2\pi r\sin\theta\, d\theta\, dx} = \frac{4}{\pi^2 r}\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta}\theta\sin\theta\, d\theta\, dx$$

$$= \frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin^2\theta\, d\theta = \frac{1}{2} + \frac{2}{\pi^2}.$$

3560. (By the Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelope will be $\frac{1}{4}$PQ $(\cot\theta + \cot\phi)$, θ and ϕ being the angles at which PQ cuts the given curve.

Solution by Professor Nash, M.A.; Prof. Cochez; *and others.*



Let PRQ, P′RQ′ be two positions of the cutting line; then, since the area PRP′ = the area QRQ′, therefore PR = RQ in the limit, and R is a point on the envelope. Similarly R′, the middle point of P′Q′, is on the envelope, and RR′ is an element of the curve. Therefore the radius of curvature is the limit of $\frac{\mathrm{RR}'}{\psi}$, where ψ is the angle PRP′.

Now $\qquad \frac{\mathrm{P'R}}{\mathrm{PR}} = \frac{\sin\theta}{\sin(\theta-\psi)} = 1 + \psi\cot\theta$, in limit;

also $\qquad \frac{\mathrm{Q'R}}{\mathrm{QR}} = \frac{\sin\phi}{\sin(\phi+\psi)} = 1 - \psi\cot\phi$ in limit,

therefore $\qquad 2\mathrm{RR'} = \mathrm{P'R} - \mathrm{Q'R} = \mathrm{PR}\,\psi(\cot\theta + \cot\phi)$,

therefore radius of curvature $= \frac{1}{4}$PQ $(\cot\theta + \cot\phi)$.

5818. (By F. C. WACE, M.A.)—If P_1, P_2, P_3, be the three forces which, acting along the surface of the plane at inclinations of β_1, β_2, β_3 to the line of greatest slope, will support the same weight, prove that

$$P_1 \cos \beta_1 (P_2^2 - P_3^2) + P_2 \cos \beta_2 (P_3^2 - P_1^2) + P_3 \cos \beta_3 (P_1^2 - P_2^2) = 0.$$

Solution by the REV. G. H. HOPKINS, M.A.; R. E. RILEY, B.A.; *and others.*

If W, α, μ be respectively the weight of the body, the inclination of the line of greatest slope to the horizon, and the coefficient of friction (F), then $F = \mu W \cos \alpha$. It is noticeable that F is the same for any supporting force, its inclination to the line of greatest slope varying with the supporting force, hence we have the usual relations

$$\mu^2 W^2 \cos^2 \alpha = P_1^2 + W^2 \sin^2 \alpha + 2P_1 W \sin \alpha \cos \beta_1 \text{ ....(1)},$$
$$\mu^2 W^2 \cos^2 \alpha = P_2^2 + W^2 \sin^2 \alpha + 2P_2 W \sin \alpha \cos \beta_2 \text{ ....(2)},$$
$$\mu^2 W^2 \cos^2 \alpha = P_3^2 + W^2 \sin^2 \alpha + 2P_3 W \sin \alpha \cos \beta_3 \text{ ....(3)}.$$

Multiplying (1), (2), (3) respectively by $P_2^2 - P_3^2$, $P_3^2 - P_1^2$, and $P_1^2 - P_2^2$, and adding, we obtain the relation

$$P_1 \cos \beta_1 (P_2^2 - P_3^2) + P_2 \cos \beta_2 (P_3^2 - P_1^2) + P_3 \cos \beta_3 (P_1^2 - P_2^2) = 0.$$

5746. (By C. LEUDESDORF, M.A.)—The circle of curvature at any point P of the curve $ay^2 = x^3$ cuts the curve in three other points: prove that the locus of the centroid of these three points, as p moves along the curve, is $3ay^2 + (3x+2a)(7x+2a)^2 = 0$.

Solution by R. A. HERMAN, B.A.; E. W. SYMONS; *and others.*

Let $y = ap^3$, $x = ap^2$; then, if (α, β) be the coordinates of the centre of the osculating circle at a point whose parameter is p, we have

$$\alpha = -\tfrac{1}{2}ap^2(2+9p^2), \quad \beta = \tfrac{4}{3}ap(1+3p^2), \quad r^2 = \tfrac{1}{36}a^2p^2(4+9p^2)^3.$$

The equation to the osculating circle is

$$\left\{X + \tfrac{1}{2}ap^2(2+9p^2)\right\}^2 + \left\{Y - \tfrac{4}{3}ap(1+3p^2)\right\}^2 = \tfrac{1}{36}a^2p^2(4+9p^2)^3 \text{ ...(1)}.$$

Let q be the parameter of the point where it meets the curve, then

$$q^6 + q^4 - \tfrac{8}{3}q^3p(1+3p^2) + q^2(2p^2+9p^4) = \tfrac{1}{3}p^4 + 2p^6.$$

This equation has 3 equal roots p; hence, dividing out by them, we get

$$q^3 + 3pq^2 + (1+6p^2)q + \tfrac{1}{3}(p+6p^3) = 0.$$

Now if (x, y) be the coordinates of the centroid of the three points where (1) meets the curve, we have

$$x = \tfrac{1}{3}a(q_1^2+q_2^2+q_3^2), \quad y = \tfrac{1}{3}a(q_1^3+q_2^3+q_3^3),$$
$$q_1^2+q_2^2+q_3^2 = 9p^2 - 2(1+6p^2) = -(2+3p^2),$$
$$q_1^3+q_2^3+q_3^3 = -(p+6p^3) - 3p\left\{9p^2 - 3(6p^2+1)\right\} = 21p^3 + 8p;$$

therefore $\qquad x = -\tfrac{1}{3}a(2+3p^2), \quad y = \tfrac{1}{3}a(8p+21p^3),$

therefore $\qquad 3x + 2a = -3ap^2, \quad 7x + 2a = -\tfrac{1}{3}a(8+21p^2);$

therefore $\qquad (7x+2a)^2(3x+2a) = -\tfrac{1}{3}a^3p^2(8+21p^2)^2 = -3ay^2;$

therefore the locus is $\quad 3ay^2 + (3x+2a)(7x+2a)^2 = 0.$

5536. (By W. H. H. HUDSON ...) $(x^2+y^2+8c^2) ...$ ... or ... and prove that its evolute is sim... ... unity,

Solution by J. HAMMOND ...

Turning the axes round ... which P and Q

$(x^2+y^2+8c ...$ ... envelope of the line

which becomes, on subtract ... $(n-m)\theta$.

$(y^2 ...$

the rationalized equation ...

Assume $x = 2c\cos^3\theta$, ... $\frac{1}{2}(n-m)\theta$.

the equation to the nor...

$(y-2 ...$

or ... $x ...$ ... $n\theta + m\sin n\theta) \ldots (1)$,

Differentiating with ...

The coordinates ... circle of radius $\frac{am}{n+m}$

tions are $x = 2c ...$ ... vertices, therefore, lie

Therefore $x+y ...$

Now put $\frac{x+}{\sqrt{2}}$ ... the equations (1) become

$... t^{-m}) + cm(t^n - t^{-n})$,

The evolut... ... $+ \mu y + \nu = 0$ meets the

rantal hype... ... in t; *i.e.*, the degree of

axes are incl...

[The figur... ... $\cos\frac{1}{2}(n-m)\theta = 0$,

BOOTH's *Geo*...

evolute is prev... $t^{-\frac{1}{2}(n+m)})$

$... + t^{-\frac{1}{2}(n-m)}) = 0$,

... y, this equation is of the ... is $n+m$.

... circular points.

3743. (By T. ... the centre of the circle to one

curve $(xy-ak)^2 = ...$

and a parallel to th...

Solution by P... ... values of t, and therefore also of ... zero values of t, which give

Let $y = x\tan\theta + a ...$

coordinates of the ext... ... infinity on this line of which only ... otherwise this line would meet the ... therefore is that given by the

Substituting in the eq...

$(ka\cos\theta + k ...$ ... the envelope and its

Dividing by k^2 and putt...

$y\cos\theta - x\sin\theta + ...$

5536. (By W. H. H. HUDSON, M.A.)—Trace the curve

$$(x^2+y^2+8c^2)^2(x^2+y^2-c^2) = 108c^2x^2y^2,$$

and prove that its evolute is similar to itself.

Solution by J. HAMMOND, M.A.; Prof. MOREL; *and others.*

Turning the axes round an angle $\frac{1}{4}\pi$, we have

$$(x^2+y^2+8c^2)^2(x^2+y^2-c^2) = 27c^2(x^2-y^2)^2,$$

which becomes, on subtracting from each side $27(x^2+y^2)^2$,

$$(x^2+y^2-4c^2)^3 = -108c^2x^2y^2,$$

the rationalized equation to the quadrantal hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}} = (2c)^{\frac{2}{3}}$.

Assume $x = 2c\cos^3\theta$, $y = 2c\sin^3\theta$; so that $dx : dy = -\cos\theta : \sin\theta$, and the equation to the normal is

$$(y-2c\sin^3\theta)\sin\theta = (x-2c\cos^3\theta)\cos\theta,$$

or $$x\cos\theta - y\sin\theta = 2c(\cos^2\theta-\sin^2\theta).$$

Differentiating with respect to θ, $x\sin\theta+y\cos\theta = 8c\sin\theta\cos\theta$.

The coordinates of the centre of curvature found by solving these equations are $x = 2c(\cos^3\theta+3\cos\theta\sin^2\theta)$, $y = 2c(3\cos^2\theta\sin\theta+\sin^3\theta)$.

Therefore $x+y = 2c(\cos\theta+\sin\theta)^3$, $x-y = 2c(\cos\theta-\sin\theta)^3$.

Now put $\dfrac{x+y}{\sqrt{2}} = \mathrm{X}$, $\dfrac{x-y}{\sqrt{2}} = \mathrm{Y}$, and $\theta = \frac{1}{4}\pi-\phi$, and there results

$$\mathrm{X} = 4c\cos^3\phi, \quad \mathrm{Y} = 4c\sin^3\phi.$$

The evolute of the quadrantal hypocycloid is therefore another quadrantal hypocycloid whose linear dimensions are twice as great, and whose axes are inclined at an angle $\frac{1}{4}\pi$ to those of the given curve.

[The figure of the quadrantal hypocycloid and its evolute is given in BOOTH's *Geometrical Methods*, Vol. I., p. 156, where this property of the evolute is proved by Boothian Tangential Coordinates.]

3743. (By T. COTTERILL, M.A.)—If a line, length k, rest upon the curve $(xy-ak)^2 = (x+b)^2(k^2-x^2)$ and the line $x = 0$, it envelopes a circle and a parallel to the four-cusped hypocycloid.

Solution by Professor NASH, M.A.; G. TORELLI; *and others.*

Let $y = x\tan\theta+\alpha$ be the equation of the variable line, then the coordinates of the extremity which lies upon the given curve are

$$x = k\cos\theta, \quad y = k\sin\theta+\alpha.$$

Substituting in the equation of the curve, we get

$$(k\alpha\cos\theta+k^2\sin\theta\cos\theta-ak)^2 = (k\cos\theta+b)^2k^2\sin^2\theta.$$

Dividing by k^2 and putting $\alpha = y-x\sin\theta$,

$$y\cos\theta-x\sin\theta+k\sin\theta\cos\theta-a = \pm(k\sin\theta\cos\theta+b\sin\theta).$$

The upper sign gives $\quad y\cos\theta-(x+b)\sin\theta-a=0$;
the envelope of which is the circle $(x+b)^2+y^2=a^2$. The lower sign gives

$$y\cos\theta-(x-b)\sin\theta+k\sin 2\theta-a=0 \text{ ....(A).}$$

The envelope of the line $y\cos\theta-(x-b)\sin\theta+k\sin 2\theta=0$
is the four-cusped hypocycloid $(x-b)^{\frac{2}{3}}+y^{\frac{2}{3}}=(2k)^{\frac{2}{3}}$,
therefore the envelope of (A) is a parallel to this curve.

5141. (By J. J. Walker, M.A.)—Show that the axis of inflexion of the cubic

$$ax^3+by^3+3fy^2z+3hx^2y+3jzx^2+3kxy^2=0$$

is

$$(af+jk)\,x+(bj+fh)\,y+4fjz=0.$$

Solution by Prof. Evans, M.A.; Prof. Nash, M.A.; *and others.*

The equation of the Hessian of the given cubic is

$$j^2kx^3+f^2hy^3+f^2jy^2z+(bj^2-2fhj)\,x^2y+fj^2zx^2+(af^2-2fjk)\,xy^2=0 \text{...(1),}$$

therefore the equation

$$(j^2k+\mu a)\,x^3+(f^2h+\mu b)\,y^3+(f^2j+3\mu f)\,y^2z+(fj^2+3\mu j)\,zx^2$$
$$+(bj^2-2fhj+3\mu h)\,x^2y+(af^2-2fjk+3\mu k)\,xy^2=0 \text{ ....(2)}$$

represents any cubic passing through all the points of intersection of the Hessian and given cubic.

If this breaks up into linear factors, one of which is the axis of inflexion, the remaining factors must be the common tangents of the Hessian and cubic at the point $x=0$, $y=0$, which is a double point on the given curve. These tangents are represented by $fy^2+jx^2=0$, and if this is a factor of (2) we must have $\mu=fj$, and the remaining factor gives

$$(af+jk)\,x+(bj+fh)\,y+4fjz=0.$$

4323. (By Professor Wolstenholme.)—If two points P, Q describe the same circle in the same sense with velocities always in the ratio $m:n$ (m, n being whole numbers prime to each other and $n>m$), the envelope of PQ is an epicycloid whose vertices lie on the given circle and cusps on a concentric circle, whose radius is $\dfrac{n-m}{n+m}$ times that of the given circle. The class is $n+m$, and the degree $2n$, and the points of contact of tangents drawn from any point on the given circle will be corners of two regular polygons of m sides and n sides respectively. The circular points at infinity are multiple points of the nth degree, of form similar to the origin in the curve $Z^{m+n}\propto X^n$, where Z is the line at infinity and X the straight line joining it to the centre of the given circle.

If the points move in opposite senses, the envelope is an hypocycloid, of degree $2n$ and class $m+n$, the other properties being the same, but for $\frac{n-m}{n+m}$ we write $\frac{n+m}{n-m}$, and for the form of the circular points at infinity, $Z^{n-m} \propto X^m$ or $\propto Y^m$.

Solution by Professor NASH, M.A.

Let O be the centre of the circle, A a point in it at which P and Q meet, AOP$=m\theta$, AOQ$=n\theta$. Then we have to find the envelope of the line

$$x\cos\tfrac{1}{2}(n+m)\theta + y\sin\tfrac{1}{2}(n+m)\theta = a\cos\tfrac{1}{2}(n-m)\theta.$$

Differentiating with respect to θ,

$$x\sin\tfrac{1}{2}(n+m)\theta - y\cos\tfrac{1}{2}(n+m)\theta = \frac{n-m}{n+m}a\sin\tfrac{1}{2}(n-m)\theta.$$

From these equations, we get

$$x = \frac{a}{n+m}(n\cos m\theta + m\cos n\theta),\quad y = \frac{a}{n+m}(n\sin m\theta + m\sin n\theta)\ldots(1),$$

the equations of an epicycloid generated by a circle of radius $\frac{am}{n+m}$ rolling on a circle of radius $a\frac{n-m}{n+m}$, and whose vertices, therefore, lie upon the given circle.

Writing c for $\frac{a}{2(n+m)}$, and t for $\cos\theta + i\sin\theta$, the equations (1) become

$$x = cn(t^m + t^{-m}) + cm(t^n + t^{-n}),\quad iy = cn(t^m - t^{-m}) + cm(t^n - t^{-n}),$$

therefore, to find the points where any line $\lambda x + \mu y + \nu = 0$ meets the curve, we shall get an equation of the degree $2n$ in t; *i.e.*, the degree of the curve is $2n$.

The tangent at any point of the curve is

$$x\cos\tfrac{1}{2}(n+m)\theta + y\sin\tfrac{1}{2}(n+m)\theta + c(n-m)\cos\tfrac{1}{2}(n-m)\theta = 0,$$

or
$$x\left(t^{\frac{1}{2}(n+m)} + t^{-\frac{1}{2}(n+m)}\right) + \frac{y}{i}\left(t^{\frac{1}{2}(n+m)} - t^{-\frac{1}{2}(n+m)}\right)$$
$$+ c(n-m)\left(t^{\frac{1}{2}(n-m)} + t^{-\frac{1}{2}(n-m)}\right) = 0,$$

and if the tangent pass through a given point xy, this equation is of the $(n+m)^{\text{th}}$ degree in t, or the class of the curve is $n+m$.

To find the relations of the curve to the circular points.

The straight line $x+iy=0$, which joins the centre of the circle to one of the circular points is given by the equation

$$2cnt^m + 2cmt^n = 0.$$

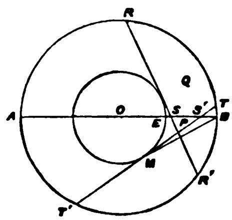
Since this is only of the n^{th} degree, n values of t, and therefore also of x and y, must be infinite; there are also m zero values of t, which give infinite values to x and y.

There are, therefore, $n+m$ points at infinity on this line of which only n lie upon the line at infinity, for otherwise this line would meet the curve in more than $2n$ points. The form therefore is that given by the equation $Z^{n+m} \propto X^n$.

When the points move in opposite directions, the envelope and its properties may be found in the same way as above.

153. NOTE ON THE SOLUTION OF QUESTION 1849. *By* E. B. SEITZ.

The solution of this question in Vol. VIII. of the *Reprint*, by Mr. S. WATSON, is incorrect, as may be seen from the following consideration: Draw the tangent BM, and let P be taken within the space BEM. Through P draw the chords RR′ and TT′ tangent to the concentric circle. Now, according to Mr. WATSON's solution, the chord through P, Q will be less than RR′ as long as Q lies within the space BSR. But if Q lies within the space BS′T, the chord through P, Q will evidently cut the concentric circle, and will consequently be greater than RR′.



By a solution entirely different from Mr. WATSON's, I find $p = \frac{1}{\pi^2}(2\beta - \sin 2\beta)^2$.

5275. (By Professor WOLSTENHOLME, M.A.)—Prove that (1) the locus of the points of intersection of circles of curvature of a parabola drawn at the ends of a focal chord is a bicircular quartic having an acnode and two double tangents; (2) the equation referred to the acnode as origin is $(x^2+y^2-2ax)^2 = 4a^2(3y^2+7x^2)$, that of the parabola being $y^2 = 4a(x+3a)$; (3) the bitangents are inclined at 30° to the axis, and the equation referred to their point of intersection is

$$(r^2 - 14ar\cos\theta + 36a^2)^2 = 4a^2r^2(4\cos^2\theta - 3),$$

the curve being its own inverse with respect to this point. [It would be difficult to show the double tangents in a figure, since for all four values of r to be real we must have $\cos\theta > \frac{1}{2}\sqrt{3}$ and $< \frac{13}{15}$, and the limiting values differ by about 5′ 50″.] (4) The curve osculates the parabola in the two points whose distance from the focus is equal to the latus rectum. (5) The equation of the curve referred to the bitangents is

$$[x^2+y^2+xy-7b(x+y)+12b^2]^2 = 4b^2xy.$$

[Writing b for $a\sqrt{3}$, so that, when $y = 0$, $(x-3b)^2(x-4b)^2 = 0$, and if we take $x = \frac{7}{2}b$, we have $(y^2 - \frac{7}{2}by - \frac{1}{4}b^2)^2 = 14b^3y$, and one value of y is nearly $\frac{1}{16}b \div (14 - \frac{7}{4}) = \frac{1}{196}b$. Thus the area included between the curve and a bitangent must be almost invisible on any ordinary scale.] (6) Two circles of curvature touch each other and also this locus at a point whose distances from the node and the centre of inversion are as 5 : 8, being $\frac{10a}{\sqrt{11}}$ and $\frac{16a}{\sqrt{11}}$.

Solution by Professor NASH, M.A.

1. The coordinates of any point on the parabola $y^2 = 4ax$ are $\frac{a}{m^2}$, $\frac{2y}{m}$; therefore the equation of the circle of curvature is

$$\left(x-2a-\frac{3a}{m^2}\right)^2+\left(y+\frac{2a}{m^3}\right)^2=4a^2\left(\frac{1+m^2}{m^2}\right),$$

or $$m^4(x^2+y^2-4ax)-6axm^2+4aym-3a^2=0,$$

which we may write $\quad Am^4+Bm^2+Cm+D=0.$

The equation of the circle of curvature at the other extremity of the focal chord through (m) is $\quad Dm^4-Cm^3+Bm^2+A=0.$

Eliminating m between these equations, we get for the locus

$$(A+B+D)(A-D)^2+AC^2=0;$$

i.e., $(x^2+y^2-10ax-3a^2)(x^2+y^2-4ax+3a^2)^2+16a^2y^2(x^2+y^2-4ax)=0.$

This equation contains the factor $x^2+(y-a)^2$, giving as part of the locus a pair of imaginary straight lines through the focus of the parabola. Dividing by $x^2+(y-a)^2$, we obtain

$$(x^2+y^2)^2-2a(x^2+y^2)(8x-9a)+3a^2(16x^2-24ax-9a^2)=0.$$

Putting $y=0$, and solving the biquadratic equation, we find

$x=3a\,.\,3a\,.\,(5\pm2\sqrt{7})\,a$, therefore $3a\,.\,0$ is a double point.

2. Transforming to this point at origin, we obtain the equation

$$(x^2+y^2-2ax)^2=4a^2(3y^2+7x^2).$$

The curve represented by this equation is a bicircular quartic, and, the tangents at the origin being given by the equation $y^2+2x^2=0$, are imaginary; therefore the origin is an acnode or conjugate point.

3. The equation may be written in the form

$$(x^2+y^2-2ax-12a^2)^2+4a^2\left\{3y^2-(x+6a)^2\right\}=0,$$

therefore the lines $3y^2-(x+6a)^2=0$ are bitangents, the points of contact lying on the circle $\quad x^2+y^2-2ax-12a^2=0.$

The bitangents make an angle $\tan^{-1}\dfrac{1}{\sqrt{3}}$ or $30°$ with the axis, and the points of contact are $0,\ \pm2a\sqrt{3}$, and $-\frac{3}{2}a,\ \pm\frac{1}{2}(3a\sqrt{3})$.

Transforming to the origin $(-6a,\,0)$, the equation becomes

$$(x^2+y^2-14ax+48a^2)^2=4a^2(3y^2+7x^2-84ax+252a^2),$$

or $$(x^2+y^2-14ax+36a^2)^2=4a^2(x^2-3y^2),$$

or in polar coordinates

$$(r^2-14ar\cos\theta+36a^2)^2=4a^2r^2(4\cos^2\theta-3).$$

Putting for r, $\dfrac{36a^2}{r'}$, we get the same equation; therefore the curve is its own inverse with respect to the circle $r=6a$.

4. To find the points where the curve meets the parabola, put $y^2=4a(x+3a)$ in the equation referred to the acnode.

The equation becomes

$$(x^2+2ax+12a^2)^2=4a^2(7x^2+12ax+36a^2),$$

or $$(x^2+2ax+12a^2)^2-4a^2(x+6a)^2=24a^2x^2,$$

i.e., $$x^2(x^2+4ax+24a^2)=24a^2x^2,\ \text{or}\ x^3(x+4a)=0;$$

therefore the curve osculates the parabola at the two points where $x=0$, and the distance of either of these points from the focus is $4a$.

5. If x', y' be coordinates of any point referred to the bitangents

$$x=\tfrac{1}{2}\sqrt{3}\,(x'+y'),\quad y=\tfrac{1}{2}(y'-x');$$

therefore $\qquad x^2+y^2 = x'^2+x'y'+y'^2,\ x^2-3y^2 = 3x'y'.$

Substituting in the equation in paragraph (3),

$$\{x'^2+x'y'+y'^2-7a\sqrt{3}\,(x'+y)+36a^2\}^2 = 12a^2x'y'.$$

Suppressing accents, and writing b for $a\sqrt{3}$, this becomes

$$\{x^2+xy+y^2-7b\,(x+y)+12b^2\}^2 = 4b^2xy.$$

6. If two circles of curvature touch, the contact must be internal, and the distance between the centres must be equal to the difference of the radii; therefore

$$\left(2a+\frac{3a}{m^2}-2a-3am^2\right)^2+\left(\frac{2a}{m^3}+2am^3\right)^2 = 4a^2(1+m^2)^3\left(1-\frac{1}{m^3}\right)^2.$$

Solving this equation, we get

$$(m^2+1)^3(3m^2-8m+3) = 0,\ \text{therefore } m = \tfrac{1}{3}(4\pm\sqrt{7}).$$

When the circles touch, the point of contact divides the line joining the centres externally in the ratio of the radii, *i.e.*, in the ratio $1 : m^3$.

From this we obtain for the coordinates of the point of contact

$$x = \frac{13}{11}a,\quad y = \pm\frac{10\sqrt{7}}{11}a.$$

If r_1, r_2 be distances from node and centre of inversion

$$r_1^2 = \left(3a-\frac{13a}{11}\right)^2+\frac{700a^2}{11^2} = \frac{1100a^2}{11^2},\ \text{therefore } r_1 = \frac{10a}{\sqrt{11}};$$

$$r_2^2 = \left(3a+\frac{13a}{11}\right)^2+\frac{700a^2}{11^2} = \frac{2816a^2}{11^2} = \frac{256a^2}{11},\ \text{therefore } r_2 = \frac{16a}{\sqrt{11}}.$$

5578. (By J. Hammond, M.A.)—Show (1) that the sum to $2m+1$ terms of the series

$$\frac{1}{n}-\frac{2m+n}{n(n+1)}+\frac{(2m+n)(2m+n-1)}{n(n+1)(n+2)}-\frac{(2m+n)(2m+n-1)(2m+n-2)}{n(n+1)(n+2)(n+3)}+\&c.$$

is $(m+n)^{-1}$; and (2) find the sum to infinity when both m and n are fractional, as also $2m+n$.

Solution by Prof. Nash, M.A.; Prof. Evans, M.A; *and others.*

The $(2+1)^{\text{th}}$ term of the series $= u_{2+1}$

$$= (-)^2\frac{(2m+n)(2m+n-1)\ \dots\ (2m+n-2+1)}{n(n+1)\ \dots\ n+2}$$

$$= \frac{(-)^2}{2(m+n)}\left\{\frac{(2m+n)(2m+n-1)\dots(2m+n-2+1)}{n(n+1)\dots(n+2-1)}+\frac{(2m+n)(2m+n-1)\dots(2m+n-2)}{n(n+1)\dots(n+2)}\right\};$$

therefore $\int_{2m+1} = \frac{1}{n} - \frac{1}{2(m+n)}\left\{\frac{2m+n}{n} - \frac{(2m+n)\ldots n}{n(n+1)\ldots(2m+n)}\right\}$

$$= \frac{1}{n} - \frac{m}{n(m+n)} = \frac{1}{m+n}.$$

If m and n be fractional, and $2m = n$, the $(2n+p+1)^{\text{th}}$ term is

$$(-)^p \frac{2n(2n-1)\ldots(-p)}{n(n+1)\ldots(3n+p)} = (-)^p \frac{2n!\,(-)^p p!\,n-1!}{3n+p!},$$

which becomes indefinitely small when p becomes indefinitely large.

$$\int_\infty = \frac{1}{n} - \frac{1}{2(m+n)} \cdot \frac{2m+n}{n} = \frac{1}{n} - \frac{2n}{3n^2} = \frac{1}{3n}.$$

5062. (By W. H. H. Hudson, M.A.)—Trace the curves

$$2\left\{x^2-(y-a)^2\right\}\left\{4x^2-(y-2a)^2\right\} + 9y^2\left\{x^2+a(y-a)\right\} = 0 \quad \ldots\ldots(1),$$

$$a(a^2-\beta\gamma)(\beta+\gamma) = \beta^2\gamma^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Solution by Professor Nash, M.A.

1. Solving the equation as a quadratic in x^2, we get

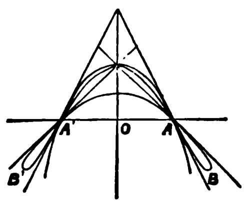
$$16x^2 = y^2 - 24ay + 16a^2 \pm y\left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}}$$

therefore x^2 will be real only when y lies between $\frac{8a}{6\sqrt{2}-3}$, and $-\frac{8a}{6\sqrt{2}+3}$. Putting $x = 0$, we get

$$2(y-a)^2(y-2a)^2 + 9ay^2(y-a) = 0,$$

or $\quad (y-a)(2y-a)(y^2+8a^2) = 0$;

therefore the curve cuts the axis of y at the points $y = \frac{a}{2}$, $y = a$. When $y > a$, all four values of x are imaginary; when $y < a > \frac{a}{2}$,



two are real, and two imaginary; when $y < \frac{a}{2} > -\frac{8a}{6\sqrt{2}+3}$, all are real; when $y < -\frac{8a}{6\sqrt{2}+3}$, all are imaginary.

The curve has no real infinite branches, since the terms of the fourth degree $8x^4 - x^2y^2 + y^4$ have no real factors. Differentiating,

$$32x\frac{dx}{dy} = 2y - 24a \pm \left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}} \pm \frac{24ay - 63y^2}{\left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}}};$$

therefore $\frac{dx}{dy} = \infty$, when $x = 0$, or $y = -\frac{8a}{6\sqrt{2}+3}$; and $= -1$ or $-\frac{1}{2}$, when $x = a$, $y = 0$; this point is therefore a double point.

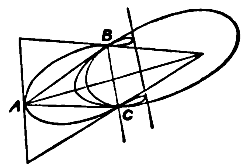
The figure is symmetrical with respect to the axis of y, and the lines

$x = \pm (y-a)$, $2x = \pm (y-2a)$, and the parabola $x^2+a(y-a)=0$, determine the compartments within which it must lie.

In the figure A, A′ are nodes, BB′ the bitangent parallel to the axis of x. At the nodes the tangents have contact of the first and second orders respectively.

2. The curve intersects the line at infinity $\alpha+\beta+\gamma=0$ in imaginary points, and therefore has no infinite branches.

The vertices B and C of the triangle of reference are nodes; the tangents at B are $\gamma=0$, $\gamma+\alpha=0$, for when $\gamma=0$, $\alpha^3\beta=0$; and three consecutive points lie on the tangent, which, however, is only a single tangent, since B is a node.



$\gamma+\alpha=0$ meets the curve in four consecutive points, and is therefore an inflexional tangent. The line $(\beta+\gamma=0)$ meet the curve in four consecutive points at A, and therefore has contact of the third order with it.

Any line $\alpha = k(\beta+\gamma)$ parallel to $\alpha=0$ will meet the curve in four points given by the equation

$$k^3\left\{\frac{(\beta+\gamma)^2}{\beta\gamma}\right\}^2-k\frac{(\beta+\gamma)^2}{\beta\gamma}-1=0, \text{ or } \frac{(\beta+\gamma)^2}{\beta\gamma}=\frac{1\pm(1+4k)^{\frac{1}{2}}}{2k^2}.$$

If $4k=-1$, we have $\dfrac{\beta}{\gamma}=3\pm 2\sqrt{2}$, and $4\alpha+\beta+\gamma=0$ is a bitangent. If $4k<-1$ all the points are imaginary. If $4k>-1$, and less than the value determined by the real root (different from zero) of the equation $\dfrac{1\pm(1+4k)^{\frac{1}{2}}}{2k^2}=4$, all the roots are real, and for greater values two are real and two imaginary. For this value of k two of the points coincide, and are situated on the line $\beta=\gamma$.

The lines $\alpha=0$, $\beta+\gamma=0$, and the conic $\alpha^2-\beta\gamma=0$, determine the compartments within which the curve lies.

5792. (By the Rev. Dr. HAUGHTON, F.R.S.)—Show that, within the limits of NEWTON's construction for the disturbing force, the *exact equation* of the surface of an equatorial canal of constant width and depth, attracted by the moon always in the equator, is $\rho = -a\cos 2\phi + b$; also trace the form of the curve for the following particular cases:—(1) $b=0$; (2) $b<a$; (3) $b=a$; (4) $b>a$, but $<5a$; (5) $b=$ or $>5a$.

I. *Hints towards a Solution; by the* PROPOSER.

1. My coordinates (ρ, ϕ) are the common (r, θ) coordinates, and do not refer to the radius of curvature and angle made by it with a fixed line.

2. My solution is *exact* (within the limits of NEWTON's construction for

the disturbing force), that is, to terms of the first order of $\left(\frac{a}{\mathrm{R}}\right)$, where a is the radius of the earth and R the distance of the moon.

3. The solution of my problem requires the conception of a *Platonic form*, like that of a river, where the water, at each several place along the bank, always behaves in the same way peculiar to that place, although the rain-drops composing it are always different.

4. The Newtonian construction for the tangential disturbing force is expressed by the equation $f = -k \sin 2\phi$, where $k = \frac{1}{400000}$ ft. per second.

II. *Solution by* Professor TOWNSEND, F.R.S.

Denoting by z the depth and by v the velocity of the water at any point (r, θ) of the surface of the canal (the coordinates r and θ being supposed measured, the former from the centre of the earth, and the latter from its line of connection with that of the moon, in the direction of the earth's rotation); by z_0, v_0, and r_0 the mean values of z, v, and r; by m the mass of the moon, by d its distance from the earth's centre, and by T the tangential component of its disturbing action at $r\theta$, which, within the limits of NEWTON's construction,

$$= -\frac{3}{2}\frac{m}{d^3}r_0 \sin 2\theta\,;$$

then, since approximately, within the limits in question,

$$vdv = \mathrm{T}ds = -\frac{3}{2}\frac{m}{d^3}r_0^2 \sin 2\theta\, d\theta,$$

and therefore, by integration,

$$v^2 = v_0^2 + \frac{3}{2}\frac{m}{d^3}r_0^2 \cos 2\theta,$$

and since, to the same degree of approximation, from the incompressibility of the water,

$$vz = \text{const.} = v_0 z_0,$$

and therefore, by substitution in the preceding equation,

$$\frac{v_0^2 z_0^2}{z^2} = v_0^2 + \frac{3}{2}\frac{m}{d^3}r_0^2 \cos 2\theta\,;$$

herefore approximately, within the same limits,

$$z = z_0\left[1 - \frac{3}{4}\frac{m}{d^3}\frac{r_0^2}{v_0^2}\cos 2\theta\right]$$

$$= z_0\left[1 - \frac{3}{4}\frac{m}{d^3}\frac{1}{\omega^2}\cos 2\theta\right],$$

where ω is the velocity of the earth's rotation, and consequently

$$r = r_0 - \frac{3}{4}\frac{m}{d^3}\frac{z_0}{\omega^2}\cos 2\theta\,;$$

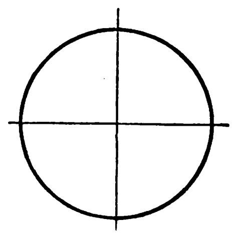


FIG. 1.

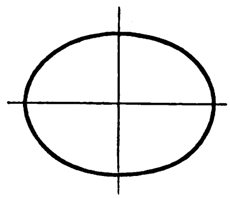


FIG. 2.

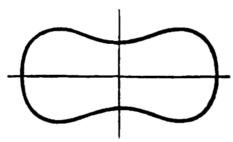


FIG. 3.

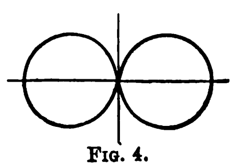


FIG. 4.

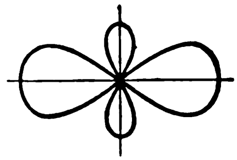


Fig. 5.

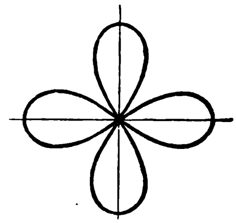


Fig. 6.

which is Dr. Haughton's equation, and exact within the limits of Newton's construction.

The tracing of the biaxal curve

$$r = a + b \cos 2\theta,$$

for different relative values of a and b, presents several interesting varieties of form, of which the following are the principal:—For $b = 0$ it is manifestly a circle of radius a (Fig. 1), respecting which, of course, nothing further need be noticed. For b small in comparison with a, as is the case in Dr. Haughton's equation, it is a biaxal oval of semi-axes $a + b$ and $a - b$, somewhat resembling an ellipse in appearance (Fig. 2), which form it retains in general character during the increase of the ratio of b to a, until that ratio becomes $= \frac{1}{5}$. For $b > \frac{1}{5}a$ but $< a$, the oval changes into the dumb-bell form (Fig. 3), and retains it in general character during the increase of the ratio from $\frac{1}{5}$ to 1. For $b = a$ the curve assumes the figure of eight form (Fig. 4), passing cuspidally twice through its centre, and touching cuspidally twice its then evanescent second axis of figure. For $a < b$ but > 0, it assumes the corolla form (Fig. 5), passing four times without inflexion through its centre, and touching at each passage one of two central tangents symmetrically situated with respect to both its axes of figure. And, finally, for $a = 0$, the four loops become equal (Fig. 6), the corolla becomes perfect, and the two common tangents at the centre to its four petals intersect at right-angles and bisect the angles between the axes of the figure. The above obviously exhaust all the real varieties of form of the curve.

III. *Solution by* J. J. Walker, M.A.

Let P be any particle of the water, T the earth's centre, TA a fixed equatorial radius, S the moon's centre, and M her accelerating force at unit distance; also let $\angle$STA $= \alpha$ (a function of the time), $\angle$PTA $= \phi$, PT $= \rho$. The difference of the components of Moon's force on P and T in the direction TP, and perpendicular to PT, respectively, are

$$-k\frac{a}{\mathrm{R}}, \text{ and } k\frac{3a \sin 2(\alpha - \phi)}{2\mathrm{R}}, \text{ where } k = \frac{\mathrm{M}}{\mathrm{R}^2}.$$

Write gravity $= -kg'$. Then, if p is the hydrodynamic pressure at P,

$$\frac{dp}{d\rho} + \frac{d^2\rho}{dt^2} = -k\left(g' + \frac{a}{\mathrm{R}}\right), \quad \frac{dp}{d\phi} + \frac{a^2 d^2\phi}{dt^2} = k\frac{3a^2 \sin 2(\alpha - \phi)}{2\mathrm{R}}.$$

Neglecting $\frac{d^2\rho}{dt^2}$ and $\frac{a^2d^2\phi}{dt^2}$ as inconsiderable (or, the latter being disregarded, and the former being treated, as by Euler, as proportional to $a - \rho$, $\frac{dp}{d\rho}d\rho + \frac{dp}{d\phi}d\phi$ would be an exact differential), and multiplying by $d\rho$ and by $d\phi$ respectively, adding and integrating, t being treated as constant, $p + \mathrm{C} = k\left\{-\left(g' + \frac{a}{\mathrm{R}}\right)\rho + \frac{3a^2 \cos 2(\alpha - \phi)}{4\mathrm{R}}\right\}$.

For the instantaneous free surface put $p = C'$, and the equation is of the form given in the question.

As regards the form of the curve $\rho = -a\cos 2\phi + b$.

1. This case has been often discussed. It is of the class of "Flores Geometrici," having four equal loops like petals radiating from the pole.

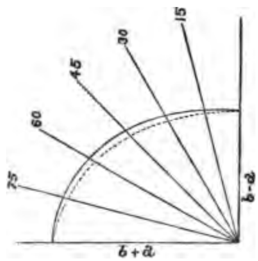
2. Also a curve of two pairs of loops with axes $\pm(a \mp b)$. In these two cases the curve passes four times through the pole without inflexion.

3. An 8-shaped curve, the pole being a double cuspidal point—the prime radius, the cuspidal tangent.

4. A dumb-bell shaped curve, having axes $= 2(b \pm a)$. It is convex towards the pole at the ends of the less, concave at the ends of the greater axis. The directions of the real points of inflexion are given by

$$2\cos^2\phi = -m + 1 + (m^2+2)^{\frac{1}{2}},$$

where m lies between 1 and 5, between which limits the dexter lies between 1 and 0; *e.g.*, when $m = \frac{7}{4}$, $\phi = \frac{1}{12}\pi$ nearly. For values of m greater than this, ϕ diminishes rapidly.



5. There is no real inflexion, and the figure differs but little from that of an ellipse with the same semi-axes $(b+a,\ b-a)$, being somewhat flatter at the ends of the axes, and consequently lying at other points outside the ellipse. A trace of a quadrant of the figure for the value $b = 6a$ is annexed, the inner dotted line being the contour of the coaxal ellipse.

5748. (By J. J. WALKER, M.A.)—If xyz satisfy $\alpha x + \beta y + \gamma z = 0$, and $u = 0$, u being a ternary form of the order p, show how to transform

$$\beta\gamma^2 y\left\{\frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2 + \frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2 - 2\frac{d^2u}{dx\,dy}\frac{du}{dx}\frac{dx}{dy}\right\}$$
$$+\beta^2\gamma z\left\{\frac{d^2u}{dx^2}\left(\frac{du}{dz}\right)^2 + \frac{d^2u}{dz^2}\left(\frac{du}{dx}\right)^2 - 2\frac{d^2u}{dz\,dx}\frac{du}{dz}\frac{du}{dx}\right\}$$

into the symmetrical form

$$\alpha\beta\gamma\left\{(p-1)\frac{du}{dx}\frac{du}{dy}\frac{du}{dz} + x\frac{d^2u}{dy\,dz}\left(\frac{du}{dx}\right)^2 + y\frac{d^2u}{dz\,dx}\left(\frac{du}{dy}\right)^2 + z\frac{d^2u}{dz\,dy}\left(\frac{du}{dz}\right)^2\right\}.$$

Solution by the PROPOSER; A. BUCHHEIM, PH.D.; *and others.*

Writing l, m, n for $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, and a, b, c, f, g, h for $\frac{d^2u}{dx^2} \dots \frac{d^2u}{dx\,dy}$, since

$$xl + ym + zn = 0,$$
$$yam^2 + ybl^2 - 2yhlm = -xalm - zamn + ybl^2 + xhl^2 + zhnl - yhlm$$
$$= -(p-1)\,l^2m + zglm + zamn + (p-1)\,l^2m - zfl^2 + zhnl$$
$$= -z\,\{amn + l\,(fl - gm - hn)\}.$$

Similarly,

$$zan^2 + zcl^2 - 2zgnl = -y\left\{amn + l\,(fl - gm - hn)\right\}.$$

The expression to be transformed is therefore equal to

$$-\beta\gamma\,(\gamma z + \beta y)\left\{amn - l\,(-fl + gm + hn)\right\},$$

or $$\alpha\beta\gamma\left\{(ax + hy + gz)\,mn + xfl^2 - xglm - zgmn - xhnl - yhmn\right\},$$

or $$\alpha\beta\gamma\left\{(p-1)\,lmn + xfl^2 + ygm^2 + zhn^2\right\}.$$

3633. (By the Rev. G. H. Hopkins, M.A.)—From any point on the surface of a circular cylinder planes are drawn; find the equation to the surface upon which the foci of all the elliptic sections are placed, and prove that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the logocyclic curve.

Solution by Professor Nash, M.A.

Let the equation of the cylinder be $x^2 + y^2 = 2ax$, and the equation of any plane through the origin $lx + my + nx = 0$ ....(1). The major axis of the section of the cylinder by the plane is the intersection of (1) and $\frac{x-a}{l} = \frac{y}{m}$, therefore the equations of the axis are

$$\frac{x-a}{l} = \frac{y}{m} = \frac{lx + my - al}{l^2 + m^2} = -\frac{nz + al}{l^2 + m^2},$$

and the lengths of the axes are $\left(\frac{2a}{n},\ 2a\right)$; hence the coordinates of the foci are given by $\frac{x-a}{nl} = \frac{y}{lm} = -\frac{z + aln^{-1}}{l^2 + m^2} = \pm\frac{a}{n}$.

Eliminating l, m, n, we get

$$\frac{z\,(2ax - x^2 - y^2)^{\frac{1}{2}}}{a} \pm (x-a) = \mp\frac{(x-a)^2 + y^2}{a},$$

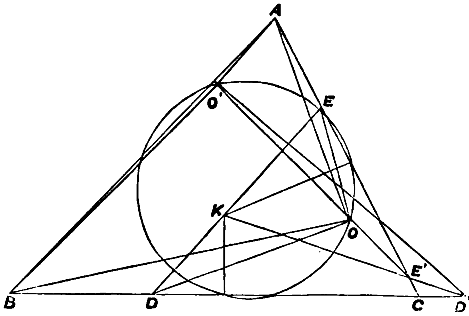
or $$z^2\,(2ax - x^2 - y^2) = \left\{y^2 + x\,(x-a)\right\}^2.$$

The section of this surface by the plane $y = 0$, which contains the axis of the cylinder, and the given point is $z^2(2a - x) = x\,(x - a,)^2$ which is the equation of the logocyclic curve.

5834. (By J. J. Walker, M.A.)—Referring to Quest. 4372 (*Reprint*, Vol. XXIII., p. 30), show that, if OD, OE, OF, perpendicular respectively to OA, OB, OC, meet BC, CA, AB respectively in D, E, F (three collinear

points, as Prof. KELLAND has shown); then, if O is a point on the circumference passing through the middle points of BC, CA, AB, the transversal DEF passes through the centre of the circle circumscribing ABC.

Solution by the PROPOSER.



If α, β, γ are vectors from O to the points A, B, C, and ω the vector to the centre of the circumference through ABC, and if $\gamma = l\alpha + m\beta$, then (Quest. 5774), if $a = T(\beta-\gamma)$, $b = T(\gamma-\alpha)$, $c = T(\alpha-\beta)$; $S_1 = S(\gamma-\alpha)(\alpha-\beta)$, $S_2 = S(\alpha-\beta)(\beta-\gamma)$, $S_3 = S(\beta-\gamma)(\gamma-\alpha)$,

$$(a^2S_1 + b^2S_2 + c^2S_3)\,\omega = (a^2S_1 + lc^2S_3)\,\alpha + (b^2S_2 + mc^2S_3)\,\beta \ldots\ldots\ldots\ldots (1).$$

Also, if δ, ϵ are the vectors OD, OE, and $S = S\beta\gamma$, $S' = S\gamma\alpha$, $S'' = S\alpha\beta$,

$$(S'-S'')\,\delta = -lS''\alpha + (S'-mS'')\beta,\quad (S''-S)\,\epsilon = (-S+lS'')\alpha + mS''\beta \ldots (2, 3).$$

Eliminating α, β from (1), (2), (3), and equating with zero the sum of the coefficients of ω, δ, ϵ in the result,

$$(1-l-m)\,(a^2S'S''S_1 + b^2S''SS_2 + c^2SS'S_3) = 0.$$

Rejecting $1-l-m = 0$, the condition of A, B, C being collinear, the second factor, expressed in terms of α, β, $S\alpha\beta$, l, m, is equal to

$$(V\alpha\beta)^2 \left\{2lmS\alpha\beta + l(l-1)\alpha^2 + m(m-1)\beta^2\right\}\left\{(1-l^2-m^2)\,S\alpha\beta - lm(\alpha^2+\beta^2)\right\};$$

of which factors, the vanishing of the first is the condition that OA, OB shall be in the same straight line; of the second that O shall be in the circumference through ABC (and thus the Quaternion Solution of 4372 is completed); while the vanishing of the third is the condition that O shall be in the circumference passing through the middle points of BC, CA, AB (see Question 5803). The annexed figure represents two positions of O (O, O′) on the nine-point circumference, and shows the collinearity of the points D, E (or D′, E′) with K the centre of the circumference through A, B, C.

5793. (By Professor SYLVESTER, F.R.S.)—

If $u_i = \left\{\left(\frac{d}{dx}\right)^i f(x)\right\}^2 - \frac{A+Bi-B}{A+Bi}\left\{\left(\frac{d}{dx}\right)^{i-1} f(x) \cdot \left(\frac{d}{dx}\right)^{i+1} f(x)\right\}$,

prove that, when $u_i = 0,\ u_{i+1} = 0,\ \ldots,\ u_{i+w} = 0$,

$$\frac{\left(\frac{d}{dx}\right)^w u_i}{u_{i+w}} = \frac{A+B(i+w)}{A+Bi}\;\frac{\left(\frac{d}{dx}\right)^i f(x)}{\left(\frac{d}{dx}\right)^{i+w} f(x)}.$$

Solution by ROBERT RAWSON.

Put D for $\left(\frac{d}{dx}\right)$, and v for $f(x)$; then, assuming the property in the question to be true for w, it can be shown to be true for $w+1$, as follows: By the question, we have

$$u_{i+w} = (D^{i+w}v)^2 - \frac{A+B(i+w-1)}{A+B(i+w)} \cdot D^{i+w-1} \cdot vD^{i+w+1} \cdot v \text{......(1)},$$

$$u_{i+w+1} = (D^{i+w+1} \cdot v)^2 - \frac{A+B(i+w)}{A+B(i+w+1)} \cdot D^{i+w} \cdot vD^{i+w+2} \cdot v \text{ ......(2)}.$$

When $u_{i+w} = 0$, then

$$(D^{i+w} \cdot v)^2 = \frac{A+B(i+w-1)}{A+B(i+w)} D^{i+w-1} \cdot vD^{i+w+1} \cdot v \text{ ....(3)}.$$

Differentiating (1) with respect to x, and using equation (3), we have

$$Du_{i+w} = 2D^{i+w}vD^{i+w+1} \cdot v - \frac{A+B(i+w-1)}{A+B(i+w)}$$
$$\times (D^{i+w}vD^{i+w+1}v + D^{i+w-1}vD^{i+w+2}v)$$
$$= \frac{A+B(i+w+1)}{A+B(i+w)} D^{i+w}vD^{i+w+1}v - \frac{A+B(i+w-1)}{A+B(i+w)} D^{i+w-1}vD^{i+w+2}v$$
$$= \frac{A+B(i+w+1)}{A+B(i+w)} \cdot \frac{D^{i+w}v}{D^{i+w+1}v}$$
$$\left\{ (D^{i+w+1}v)^2 - \frac{A+B(i+w)}{A+B(i+w+1)} D^{i+w}vD^{i+w+2}v \right\} \text{ ...(4)}.$$

From (2) and (4), $$\frac{Du_{i+w}}{u_{i+w+1}} = \frac{A+B(i+w+1)}{A+B(i+w)} \cdot \frac{D^{i+w} \cdot v}{D^{i+w+1} \cdot v} \text{..................(5)}.$$

Assume $$D^w \cdot u_i = \frac{A+B(i+w)}{A+Bi} \cdot \frac{D^i \cdot v}{D^{i+w} \cdot v} u_{i+w} \text{ ....(6)};$$

differentiate (6), and put $u_{i+w} = 0$, then

$$D^{w+1} \cdot u_i = \frac{A+B(i+w)}{A+Bi} \cdot \frac{D^i \cdot v}{D^{i+w} \cdot v} \cdot Du_{i+w} \text{ ....(7)}.$$

From (7) and (5), $$\frac{D^{w+1} \cdot u_i}{u^{i+w+1}} = \frac{A+B(i+w+1)}{A+Bi}, \frac{D^i \cdot v}{D^{i+w+1} \cdot v} \text{...............(8)}.$$

Hence, if the property be true for w, it is true for $w+1$. The property is true when $w=0$, then it is true for $w=1$, &c.

5852. (By Professor MINCHIN, M.A.)—Any rigid body is kinetically equivalent to three equal uniform spheres whose centres are the vertices of a maximum triangle inscribed in a certain ellipse lying in one of the principal planes at the centre of gravity of the body. [From WOLSTENHOLME's *Problems*, 2nd ed., Ex. 2601.]

Prove this geometrically from the fact that, if two intersecting rectangular lines of constant lengths revolve in a fixed plane about their point

of intersection, a point whose abscissa and ordinate are the projections of these lines on any fixed line, will trace out an ellipse.

Solution by the PROPOSER.

Let the principal radii of gyration at the centre of gravity O of the body be k_1, k_2, k_3, and let three uniform spheres, the mass of each being one-third of that of the body, have their centres at the points $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$, $(x_3\, y_3\, z_3)$, the radius of each being r. (The axes of coordinates are the principal axes of the body.)

Then the conditions for kinetic equivalence are

$$x_1+x_2+x_3=0,\quad y_1+y_2+y_3=0,\quad z_1+z_2+z_3=0 \;\ldots\ldots\ldots\ldots(1);$$

$$x_1y_1+x_2y_2+x_3y_3=0,\quad y_1z_1+y_2z_2+y_3z_3=0,\quad z_1x_1+z_2x_2+z_3x_3=0\ldots\;(2);$$

$$\left.\begin{array}{l}\frac{3}{5}r^2+x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2=3k_3^2,\\ \frac{3}{5}r^2+x_1^2+x_2^2+x_3^2+z_1^2+z_2^2+z_3^2=3k_2^2,\\ \frac{3}{5}r^2+y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2=3k_1^2\end{array}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Assume three points, P, Q, R, whose coordinates are respectively $(x_1\, x_2\, x_3)$, $(y_1\, y_2\, y_3)$, $(z_1\, z_2\, z_3)$. Then equations (1) show that these points all lie in a plane equally inclined to the axes of coordinates, the cosine of the angles of inclination being $\surd\frac{1}{3}$. Also, equations (2) show that the three lines OP, OQ, OR (which are coplanar, observe) are mutually rectangular. This is impossible unless one of them vanishes. Suppose $\mathrm{OR}=0$; that is, $z_1=z_2=z_3=0$. Hence the centres of the spheres all lie in the plane xy.

Again, equations (3) give $\frac{3}{5}r^2=3\,(k_1^2+k_2^2-k_3^2)$,

and also $\qquad \mathrm{OP}^2=3\,(k_3^2-k_1^2),\quad \mathrm{OQ}^2=3\,(k_3^2-k_2^2).$

Now x_1 and y_1 are the projections of OP and OQ along the axis of x; and a point, whose coordinates are the projections of OP and OQ along a line making an angle θ with the plane in which OP and OQ revolve, describes the ellipse

$$\frac{x^2}{\mathrm{OP}^2}+\frac{y^2}{\mathrm{OQ}^2}=\cos^2\theta.$$

Hence, since $\cos^2\theta=\frac{2}{3}$ in this case, $\dfrac{x_1^2}{k_3^2-k_1^2}+\dfrac{y_1^2}{k_3^2-k_2^2}=2,$

together with the very same equation for $(x_2\, y_2)$ and $(x_3\, y_3)$; and equations (1) show that the centres are the vertices of a triangle whose centroid is the centre of mass of the body.

5387. (By Professor CAYLEY, F.R.S.)—Show that a cubic surface has at most 4 conical points, and a quartic surface at most 16 conical points.

Solution by J. HAMMOND, M.A.

By reasoning precisely similar to that in SALMON'S *Higher Curves*, we see that the degree of the reciprocal must be diminished by 2 for each

conical point on the surface. Hence the degree of the reciprocal of a surface of degree m, having 8 conical points, is $m(m-1)^2-28$. [See SALMON's *Geometry of Three Dimensions*, Art. 275, 2nd ed.] Thus a cubic with 4 conical points and a quartic with 16 conical points reciprocate into quartics with 4 and 16 singular tangent planes respectively. And they cannot have 5 conical points in the case of the cubic, or 17 in the case of the quartic; for then they would reciprocate into surfaces of the second degree.

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154. NOTE ON LANDEN'S THEOREM; *by* CHRISTINE LADD.

That part of the proof of LANDEN's Theorem which is given in paragraph 54 of Professor CAYLEY's *Elliptic Functions* can be simplified as follows:—

Reverting to the original two circles, if in the figure $\angle \mathrm{APO} = \theta$ $(= \frac{1}{2}\pi + \phi - \psi)$, then obviously $\mathrm{AA'} \cos \mathrm{MAO} = \mathrm{AM}d\theta$, that is,

$$\mathrm{R} \,.\, 2d\phi \sin \mathrm{AOG} = \mathrm{AG}\; 2d\phi = \mathrm{AM}d\theta;$$

hence the equation $\dfrac{d\phi}{\mathrm{AM}} = -\dfrac{d\psi}{\mathrm{BM}}$ may be completed into

$$\frac{d\phi}{\mathrm{AM}} = -\frac{d\psi}{\mathrm{BM}} = \frac{d\theta}{2\mathrm{AG}};$$

and, observing that $\mathrm{AG}^2 = \mathrm{AO}^2 - \mathrm{OG}^2 = \mathrm{R}^2 - (\mathrm{D} \sin\theta - r)^2$,

the equation is $$\frac{d\phi}{\Delta(k, \phi)} = \frac{d\theta\,[(\mathrm{R}+\mathrm{D})^2 - r^2]^{\frac{1}{2}}}{2\,[\mathrm{R}^2 - (\mathrm{D}\sin\theta - r)^2]^{\frac{1}{2}}}.$$

We have $\mathrm{OG} = \mathrm{R} \sin \mathrm{OAG}$; but $\angle \mathrm{OAG} = \phi + \psi - \frac{1}{2}\pi = 2\phi - \theta$;

hence $$\mathrm{D} \sin\theta - r = \mathrm{R} \sin(2\phi - \theta),$$

which is an integral equation corresponding to the above differential equation.

5849. (By Sir JAMES COCKLE, F.R.S.)—Find the general solution of the linear partial differential equation of the first order

$$\frac{n}{z}(z^m + \mathrm{A}z^r + \mathrm{B})(p+q) = \frac{n}{n-r}\left(\frac{d\mathrm{A}}{dx} + \frac{d\mathrm{A}}{dy}\right) z^r + \frac{d\mathrm{B}}{dx} + \frac{d\mathrm{B}}{dy} \ldots\ldots (1),$$

wherein n, m, and r are unequal, and A and B are free from z, and are not functions one of the other.

I. *Solution by the* PROPOSER.

Generally (taking $n > m$) let the summation S refer to r, and be made from $r = m$ to $r = 0$, and let the summa ion Σ refer to a and x, so that

$\Sigma ax = a_1x_1 + a_2x_2 + \ldots + a_lx_l$ wherein $x_1, x_2, \ldots x_l$ are independent variables, and the sum $(a_1 + a_2 + \ldots + a_l)$ of the constants vanishes, *i. e.*, $\Sigma a = 0$. Then we shall have

$$\Sigma\frac{d}{dx}\Sigma ax = \left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots + \frac{d}{dx_l}\right)\Sigma ax = \Sigma a = 0,$$

and $\Sigma\frac{dz}{dx}$ will represent $\frac{dz}{dx_1} + \frac{dz}{dz_2} + \ldots + \frac{dz}{dx_l}$.

Let $\phi\Sigma ax$ be an arbitrary function of Σax, and let

$$z^n - (\phi\Sigma ax)\left(\mathrm{S}\,\frac{n\mathrm{A}_r}{n-r}z^r\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.});$$

then, operating on (2) with $\frac{z}{n}\Sigma\frac{d}{dx}$, we have

$$\left\{z^n - (\phi\Sigma ax)\left(\mathrm{S}\,\frac{r\mathrm{A}_r}{n-r}z^r\right)\right\}\Sigma\frac{dz}{dx} - z\,(\phi\Sigma ax)\left(\mathrm{S}\,\frac{1}{n-r}\Sigma\frac{d\mathrm{A}_r}{dx}z^r\right) = 0,$$

for
$$\Sigma\frac{d}{dx}\phi\Sigma ax = 0.$$

From the result of the operation eliminate z^n by means of (2), and divide out $\phi\Sigma ax$; we shall have

$$\mathrm{SA}_r z^r\Sigma\frac{dz}{dx} - z\mathrm{S}\frac{z^r}{n-r}\Sigma\frac{d\mathrm{A}_r}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i.}),$$

and, as (ii.) is a general solution of (i.), so is

$$z^n - \phi\,(y-x)\left\{\frac{nz^m}{n-m} + \frac{n\mathrm{A}z^r}{n-r} + \mathrm{B}\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

a general solution of (1); viz., of

$$\frac{n}{z}(z^m + \mathrm{A}z^r + \mathrm{B})\ (p+q) = \frac{n}{n-r}\left(\frac{d\mathrm{A}}{dx} + \frac{d\mathrm{A}}{dy}\right)z^r + \frac{d\mathrm{B}}{dx} + \frac{d\mathrm{B}}{dy}\ldots\ldots(1).$$

5819. (By Prince Camille de Polignac.)—On each side of a triangle as segment, an internal arc of a circle is drawn containing an angle of 120°; (1) prove that the coordinates of the common point of intersection of the three circles referred to the given triangle are proportional to

$$\frac{1}{\sin\mathrm{A} + \sqrt{3}\cos\mathrm{A}},\quad \frac{1}{\sin\mathrm{B} + \sqrt{3}\cos\mathrm{B}},\quad \frac{1}{\sin\mathrm{C} + \sqrt{3}\cos\mathrm{C}};$$

and (2) find a characteristic feature of the point in relation to the triangle.

I. *Solution by the* Proposer.

1. Write the three circles $\mathrm{MN} = x^2$, $\mathrm{N'L} = y^2$, $\mathrm{L'M'} = z^2$; then, ρ, σ, τ being constant multipliers, we find, attending to data,

$$\sigma\mathrm{M} \equiv z\sin 60^\circ - x\sin(\mathrm{B}-60^\circ),\quad \tau\mathrm{N} \equiv y\sin 60^\circ - x\sin(\mathrm{C}-60^\circ);$$

or putting, for shortness' sake,

$$\sin 60^\circ = K,\quad \sin(A-60^\circ) = \alpha,\quad \sin(B-60^\circ) = \beta,\quad \sin(C-60^\circ) = \gamma,$$

$$\sigma M = Kz-\beta x,\quad \tau N' = Kx-\gamma y,\quad \rho L' = Ky-\alpha z,$$

$$\tau N = Ky-\gamma x,\quad \rho L = Kz-\alpha y,\quad \sigma M' = Kx-\beta z.$$

Take $\rho = \sin A$, $\sigma = \sin B$, $\tau = \sin C$; and it will be easily seen that M, N, &c., represent the actual lengths of the coordinates, the rule of signs being duly attended to. Writing out, we find, for the first circle,

$$(\beta\gamma-\sigma\tau)\,x^2+K^2yz-K\gamma zx-K\beta xy = 0.$$

Putting this into the form $(ln+my+nz)(\rho x+\sigma y+\tau z)+h(\rho y^2+\sigma zx+\tau xy)$, to which the equation of any circle may be brought (Salmon's *Conics*), we find

$$m = n = 0,\; l = \frac{\beta\gamma-\sigma t}{\rho},\; h = \frac{K^2}{\rho};\quad \text{whence,}$$

for the first circle, $(\beta\gamma-\sigma\tau)(\rho x+\sigma y+\tau z)\,x+K^2(y\rho z+\sigma zx+\tau xy) = 0$;
and for the other two, through a circular permutation,

$$(\gamma\alpha-\tau\rho)(\rho x+\sigma y+\tau z)\,y+K^2(\rho yz+\&c.) = 0,$$

$$(\alpha\beta-\rho\sigma)(\rho x+\sigma y+\tau z)\,z+K^2(\rho yz+\&c.) = 0.$$

The radical axes are

$$(\beta\gamma-\sigma\tau)\,x = (\gamma\alpha-\tau\rho)\,y,\; (\gamma\alpha-\tau\rho)\,y = (\alpha\beta-\rho\sigma)\,z,\; (\alpha\beta-\rho\sigma)\,z = (\beta\gamma-\sigma\tau)\,x,$$

and they meet in a point the coordinates of which can be written

$$x = \frac{1}{\beta\gamma-\sigma\tau},\; y = \frac{1}{\gamma\alpha-\tau\rho},\; z = \frac{1}{\alpha\beta-\rho\sigma}.$$

Writing at full length, we find

$$\frac{1}{x} = \sin(B-60^\circ)\sin(C-60^\circ)-\sin B\sin C$$

$$= \tfrac{1}{2}\left\{\cos(B-C)-\cos(B+C-120^\circ)-\cos(B-C)+\cos(B+C)\right\}$$

$$\frac{1}{x} = \tfrac{1}{2}-\left\{\cos A+\cos(A-60^\circ)\right\},$$

whence, on expanding and dropping constant factors, we find the result stated in the question.

The point will lie inside the triangle, coincide with a vertex, or lie outside, according as every angle $<120^\circ$, or one of them $=\;>120^\circ$.

2. The fundamental property is the following. Take any point M inside triangle ABC. Place MA, MB, MC $= \alpha, \beta, \gamma$ respectively; then $\alpha+\beta+\gamma$ is a minimum when M coincides with the point just found. This appears at once from the fact that, if an ellipse be described with B, C as foci, and a circle be drawn with A as a centre to touch the ellipse, the point of contact M will give a minimum sum, $\alpha+\beta+\gamma$, with respect to all the points of the ellipse. The same remark applies to an ellipse and circle foci C, A and centre B, or foci A, B and centre C. Consequently the minimum-minimorum occurs at a point such that each vector, α, β, or γ, bisects the angle between the other two. In other words, inside the triangle,

$$\angle BMC = \angle CMA = \angle AMB = 120^\circ.$$

We can extend the notion of minimum to the algebraical sum $\alpha+\beta+\gamma$, reckoning each vector, say α = MA, as negative, where the x of M is negative, and so on, *mutatis mutandis*. Working with hyperbolas as well as with ellipses, we shall infer one more point where each vector bisects internally or externally the angle under the other two. It will be given likewise as the intersection of three circles of which the sides of the

triangle are chords subtending 120° externally, and its coordinates are

$$x = \frac{1}{\sin A - \sqrt{3}\cos A}, \quad y = \frac{1}{\sin B - \sqrt{3}\cos B}, \quad z = \frac{1}{\sin C - \sqrt{3}\cos C}.$$

When $A = B$ the two minima-points lie on the bisector of $\angle C$. When $A = B = C = 60°$, the three external circles coincide with the circumscribing circle, and the minimum point, the limit of their intersection, depends in position on the mode of degenerescence of the triangle. But, as it must lie on the bisector of one of the angles, it can only coincide with one of the intersections of the bisectors with the circumscribing circle at which $\alpha+\beta+\gamma = 0$.

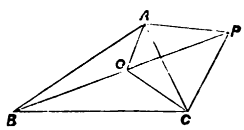
The locus of minima is simply the locus of the feet of normals drawn from a fixed point to a system of confocal conics, or, which is the same, the locus of points of contact of tangents drawn from the same point. An easy description of it is the following. For example, let A be the fixed point, and B, C the foci. Then draw through A any radius vector AR. From B let fall a perpendicular BP on it, and produce it a length $PB' = BP$. Then the line B′C meets AR in a point M, which belongs to the locus. For, obviously, M is the vertex of an isosceles triangle MBB′, the two sides of which pass through B and C respectively, while AM is the internal or external bisector of $\angle M$. This mode of description will readily disclose the chief characteristics of the curve, which is a circular cubic, as is, I suppose, well known. It will be found to pass through A, B, C with a node in A, the tangents being the two bisectors of $\angle A$. The asymptote is parallel to the bisector of the side BC drawn through A, &c. The Cartesian equation, where A is the origin, follows immediately, being, in fact, $u_2+u_3 = 0$, where u_2 = product of bisectors of $\angle A$, and $u_3 = lU$, l being the bisector of the side BC, while U is determined by the condition that the curve should pass through B and C. The trilinear equation will be found to be $S_1 \equiv 2\cos B \, . \, y^2 z - 2\cos C \, . \, z^2 y + x(y^2 - z^2) = 0$.

By permuting circularly A, B, C and x, y, z, S_2 and S_3 will be obtained, and it will appear that $S_1 x + S_2 y + S_3 z \equiv 0$,

showing that the three cubics pass through the same points of intersection, agreeably to former conclusions. These points are to be reckoned as follows:—

$$\left.\begin{array}{lr} \text{The two circular points} \;\ldots\;\ldots\;\ldots & 2 \\ \text{The three vertices} \;\ldots\;\ldots\;\ldots\;\ldots & 5 \\ \text{The two minima points} \;\ldots\;\ldots\;\ldots & 2 \end{array}\right\} \text{Total } 9.$$

II. *Solution by* JOHN YOUNG, B.A.



Let O be the point whose coordinates are required. Join AO, BO, CO, and produce BO to meet the circle COA again in P. The angles AOP, COP each contain 60°, and are respectively equal to ACP, CAP, by reason of the circle APCO; from which it follows that the triangle ACP is equilateral, hence $PC = PA = b$. Also angle $BCP = C+60°$, $BAP = A+60°$; and perpendiculars from P on BA, BC produced are respectively

$$b\sin(A+60°), \quad b\sin(C+60°).$$

Hence equation of line BO is

$$\alpha \sin (A + 60^\circ) = \gamma \sin (C + 60^\circ).$$

Similarly, equation of line CO is

$$\beta \sin (B + 60^\circ) = \alpha \sin (A + 60^\circ).$$

Thus, O is determined by the equations

$$\alpha \sin (A + 60^\circ) = \beta \sin (B + 60^\circ) = \gamma \sin (C + 60^\circ);$$

or, expanding and substituting values for sin 60°, cos 60°,

$$\alpha : \beta : \gamma = \frac{1}{\sin A + \sqrt{3} \,.\, \cos A} : \frac{1}{\sin B + \sqrt{3} \,.\, \cos B} : \frac{1}{\sin C + \sqrt{3} \,.\, \cos C}.$$

The sum of the lines OA, OB, OC is a minimum (THOMSON's *Euclid*, Appendix).

5861. (By HUGH MCCOLL, B.A.)—Required a solution of the following logical problem:—Suppose it to have been ascertained by observation (1) that, whenever the events A and B happen together, they are invariably followed by the event C, and also by either the event D or the event E; and (2) that, whenever the events D and E both happen, they have invariably been preceded by the event A, or else by both the events B and C. When may we conclude (from the occurrence or non-occurrence of the events A, B, C, D)—(1) that E will certainly happen; (2) that E will certainly *not* happen?

I. *Solution by* Prof. LLOYD TANNER, M.A.

Any combination of the occurrences and non-occurrences of A, B, C, D, E, may be represented by a symbol such as ABC′D′E, where the accents indicate non-occurrences. Write out the complete set of combinations, 32 in number; from these erase those excluded by the conditions of the problem. Of the remainder, some are in pairs, such as A′BCD′E, A′BCD′E′; whence the combination A′BCD′ tells nothing as to the occurrence of E. Besides these pairs, we have ABCD′E without the corresponding E′ symbol, and A′B′C′DE′, A′BC′DE′, A′B′CDE′ without the corresponding E symbol. Hence E will happen if A, B (and C) happen, but D does not happen; E will not happen if D happens, and either A and B or A and C do not happen. This may serve as a verification of the results otherwise obtained.

II. *Solution by the* PROPOSER.

Let the statement a assert the occurrence of the event A, with any additions or limitations as to time, place, or manner, necessary to distinguish the event A from any other event that might be confounded with it; and let a' be the denial of a, asserting the non-occurrence of A under the stated limitations. Let b, b', c, c', &c., be similarly interpreted. The

data of the problem will then be the two implications,

$$ab : c(d+e), \quad de : a+bc.$$

Solving these implications with respect to e and e', which, in this problem, may be done by mere inspection, we get

$$ab(c'+d') : e, \quad abc'+da'(b'+c') : e'.$$

The term abc', being common to both antecedents, is zero, from which we learn that ab implies c, symbolically expressed by the implication $ab : c$. Omitting this zero term, we get

$$abd' : e, \quad da'(b'+c') : e'.$$

This agrees with the result otherwise obtained by Professor TANNER.

[To anyone who has read Mr. McCOLL's second paper on the *Calculus of Equivalent Statements*, published in the *Proceedings* of the London Mathematical Society, the above solution will be perfectly clear as it stands, without a stroke of extra working.]

5832. (By the EDITOR.)—In BOOTH's *Geometrical Methods*, Vol. I., p. 114, it is shown that the sections of the surface of centres (S) by a principal plane of the ellipsoid are an ellipse and the evolute of an ellipse. Prove that the principal plane *intersects* on one sheet of the surface in the evolute section, and *touches* the other sheet in the elliptic section. And, generally, that the normals at two consecutive points of a line of curvature of the ellipsoid *intersect* on one sheet of the surface of centres, while their plane touches the other sheet.

Solution by H. STABENOW, M.A.

Since the evolute section of S by a principal plane of the ellipsoid abc, for instance by the xy-plane, is the evolute of the ellipse ab, it follows that the normals N and N′ to the latter at any two of its consecutive points, A and B, intersect on the former, and consequently on one sheet of S, "as two consecutive tangents to it;" and that their plane, that is the principal plane, necessarily intersects that same sheet.

Further, there exists generally, at each point of a given surface, a couple of principal sections, and also of lines of curvature, cutting each other at right angles; therefore—

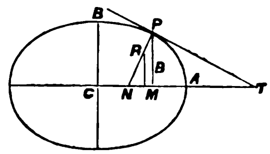
1. The elliptic section of S is the locus of the extremities of the radii of curvature at A, B, &c., of the ellipses V, V′, &c., resulting from the intersection of the ellipsoid by planes through the normals N, N′, &c., and perpendicular to the ellipse or the xy-plane.

2. Through each point, as A, of the ellipse ab, and its consecutive in the vertical section V, passes a line of curvature.

From this, and that the locus of the principal centres, answering to the different points of a line of curvature, is on the surface S a curve, to which the corresponding radii of curvature are the consecutive tangents, it is plain that any two consecutive normals, as N and N′, while each intersects the elliptic section of the surface, are at two consecutive points of this section tangents to the surface itself; and that consequently the plane of these normals, that is the principal plane, must touch the surface.

In like manner, it may be shown that the normals at two consecutive points of a line of curvature enjoy the property enunciated in the theorem.

The equation of the elliptic section of the surface S can be obtained directly as follows:—



Considering (see fig.) any point P (α, β) of the ellipse ab, and designating the conjugate of the diameter through P by b'; the distance of the centre C to the tangent PT by $p = \frac{ab}{b'}$; the sections of the ellipsoid (ellipses) by planes passing through the normal PN and the ordinate PM ($= \beta$), and perpendicular to the plane of the ellipse ab, respectively, by V and E; the radii of curvature at P of V and E respectively by PR and ρ; the angle between PN and PM, or between V and E, by γ; and the coordinates of the point R by x, y:

then since $\rho = \frac{c^2(a^2-\alpha^2)^{\frac{1}{2}}}{ab}$ and $\cos\gamma = \frac{a^2(a^2-\alpha^2)^{\frac{1}{2}}}{a^4-(a^2-b^2)\alpha^2} = \frac{(a^2-\alpha^2)^{\frac{1}{2}}}{a(a^2-e^2\alpha^2)}$,

where $e^2 = \frac{a^2-b^2}{a^2}$, we find, by a known theorem, for the value of PR,

$$\mathrm{PR} = \frac{\rho}{\cos\gamma} = \frac{c^2(a^2-e^2\alpha^2)}{ab} = \frac{c^2b'}{ab} = \frac{c^2}{p},$$

and consequently $\frac{\mathrm{PN}}{\mathrm{PR}} = \frac{\left(\frac{b^2}{p}\right)}{\left(\frac{c^2}{p}\right)} = \frac{b^2}{c} = \frac{\beta}{\beta-y} = \frac{\frac{b^2}{a^2}\alpha}{\alpha-x}$;

from which we deduce $\frac{\alpha}{a} = \frac{ax}{a^2-c^2}$, $\frac{\beta}{b} = \frac{by}{b^2-c^2}$.

Substituting these values of $\frac{\alpha}{a}$, $\frac{\beta}{b}$ in the equation of the ellipse $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$, we get, for that of the elliptic section of the surface S,

$$\frac{a^2x^2}{(a^2-c^2)^2} + \frac{b^2y^2}{(b^2-c^2)^2} = 1.$$

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155. Note on Mr. Stabenow's Solution of Question 5832.

By J. Hammond, M.A.

The first part of the solution is perfectly general, and holds for any surface whatever, and its surface of centres. It is, in fact, identical with Art. 302 of Salmon's *Geometry of Three Dimensions* (2nd Ed., p. 238). But Salmon's proof of the theorem in question is so clear, that it is worth giving in his own words.

"Now if from a point, not on a surface, be drawn two consecutive tangent lines to the surface, the plane of those lines is manifestly a tangent plane to the surface; for it is a tangent plane to the cone which is drawn from the point touching the surface. But if two consecutive tangent lines intersect on the surface, it cannot be inferred that their plane touches the

surface. For if we cut the surface by any plane whatever, any two consecutive tangents to the curve of section (which, of course, are also tangent lines to the surface) intersect on the curve, and yet the plane of these lines is supposed not to touch the surface.

"Consider now the two consecutive normals at the points M, N, these are both tangents to both sheets of the surface of centres. And since the point C, in which they intersect, is on the first sheet, but not necessarily on the second, the plane of the two normals is the tangent plane to the second sheet of the surface of centres."

In BOOTH's *Geometrical Methods* the evolute section in the plane of xy is obtained by putting $\zeta = 0$ in the tangential equation of the surface of centres. This shows that the tangent planes perpendicular to the plane of xy envelope a cylinder whose base is the evolute of the ellipse (a, b).

The elliptic section, on the other hand, is obtained by putting $\zeta = \infty$; *i. e.*, making the plane of xy a tangent plane.

Thus the plane of xy intersects the surface of centres at right angles in the evolute section, and touches it along the elliptic section.

5823. (By Professor TOWNSEND, F.R.S.)—A uniform cylindrical elastic rod, supported at both extremities and at both points of trisection by four rigid props in the same horizontal right line, being supposed in strained equilibrium, with slight deflection of form, under the action of gravity; required the distribution of its weight between the props, with the points of inflexion of its strained axis arising from their reactions.

I. *Solution by* Professor BALL, F.R.S.

Let E be the coefficient of elasticity.

Let I be the moment of inertia of a vertical section of the rod about a horizontal line drawn in the plane of section through the neutral axis.

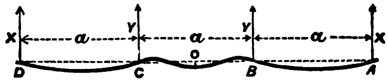
Let X be the reaction of either of the extreme props.

Let Y be the reaction of either of the intermediate props.

Let Δ be the deflection of the central point of the neutral.

Let $3a$ be the length of the axis of the entire rod.

Let w be its load per running foot.



Extreme segment.—Let x, y be the coordinates of a point on the neutral axis of the right-hand segment AB of the rod, the origin of coordinates being at the extremity A; then, from the well-known formulæ, we have

$$\mathrm{EI}\,\frac{d^2y}{dx^2} = \mathrm{X}x - \tfrac{1}{2}wx^2,$$

whence, by integration, $\mathrm{EI}y = \tfrac{1}{6}\mathrm{X}x^3 - \tfrac{1}{24}wx^4 + \mathrm{C}x + \mathrm{C}'$;

but $y = 0$ if $x = 0$ or $x = a$, whence finally,

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{X}\,(x^3 - a^2x) + \tfrac{1}{24}\,(a^3x - x^4)\,w \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\mathrm{EI}\,\frac{dy}{dx} = \mathrm{X}\,(\tfrac{1}{2}x^2 - \tfrac{1}{6}a^2) + (\tfrac{1}{24}a^3 - \tfrac{1}{6}x^3)\,w \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If $x = a$, we have $\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{\mathrm{EI}}(\tfrac{1}{3}a^2\mathrm{X} - \tfrac{1}{8}a^3w)$ .... (3).

Central segment.—As the whole weight of the rod is equal to the sum of the reactions at the points of support, we have $\mathrm{Y} + \mathrm{X} = \tfrac{3}{2}aw$.

Taking the origin of coordinates at the right-hand point of trisection B, we have, for the equation of the central segment,

$$\mathrm{EI}\frac{d^2y}{dx} = \mathrm{Y}x + \mathrm{X}(a+x) - a(\tfrac{1}{2}a + x)w - w(\tfrac{1}{2}x^2).$$

Integrating and reducing, we find

$$\mathrm{EI}y = -\tfrac{1}{2}a\mathrm{X}x + \tfrac{5}{24}a^3wx + \tfrac{1}{2}a\mathrm{X}x^2 - \tfrac{1}{4}a^2wx^2 + \tfrac{1}{12}awx^3 - \tfrac{1}{24}wx^4 \text{......(4)},$$

and, when $x = 0$, we have $\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{\mathrm{EI}}(\tfrac{5}{24}a^3w - \tfrac{1}{2}a^2\mathrm{X})$ ....(5).

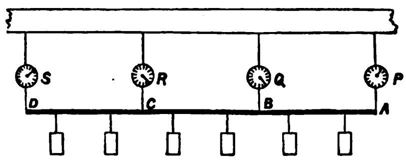
As the curvature of the neutral axis is not abruptly altered when the rod crosses the supports, we must have the values of $dy \div dx$ in the equations (3) *and* (4) *equal;* whence

$$\mathrm{P}a^2\mathrm{X} - 3a^3w = 5a^3w - 12a^2\mathrm{X}, \quad \text{or} \quad \mathrm{X} = \tfrac{2}{5}aw.$$

The entire load L being $3aw$, we have finally

$$\mathrm{X} = \tfrac{4}{30}\mathrm{L}; \quad \mathrm{Y} = \tfrac{11}{30}\mathrm{L}.$$

Experimental verification.—This result was verified by a simple experiment. A pine rod AD, four feet long and half-an-inch square, is suspended by four dynamometers P, Q, R, S in the manner shown in the figure, the distances AB, BC, CD being equal. Six weights of 14 lbs. each were hung from the beam, at equal distances apart, so as to produce approximately the effect of a continuous load, and the dynamometers were adjusted by the wire strainers from which they depended, so that the four points of support should be in a horizontal line.



The total load of the rod, including its own weight and also the weights of the iron *s*-hooks by which the load was attached, was 86 lbs. According to the theory, therefore, the strain on the extreme dynamometers should be each 11·5 lbs., and that borne by the two interior should be 31·5 lbs. The dynamometers were graduated to read to single pounds, and the two extreme ones showed 10 lbs. and 11 lbs. respectively, while the two interior ones marked 32, 33. The accordance is quite as close as could be expected. The slight discrepancy is due to the circumstance that the condition of uniform loading is only approximately fulfilled.

Transformation of the equations.—If in equation (4) we make $x = a \div 2$, and substitute for X its value $2aw \div 5$, we obtain, for the deflection at the centre, the expression $\quad \Delta = \dfrac{wa^4}{1920\mathrm{EI}}.$

Introducing this value into the equation of the central segment, *and transforming the origin of coordinates to the centre* O *of the line* AD, we

obtain finally the equation of the neutral axis

$$y = \Delta\left(2\frac{x}{a}-1\right)\left(2\frac{x}{a}+1\right)\left((20)^{\frac{1}{2}}\frac{x}{a}-1\right)\left((20)^{\frac{1}{2}}\frac{x}{a}+1\right) \ldots (6).$$

For the extreme segment of the rod with the origin still at the extremity A, we have, for the neutral axis, the equation

$$y = 16\Delta\left(3\frac{x}{a}-8\frac{x^3}{a^3}+5\frac{x^4}{a^4}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Deflection in the extreme segment.—If we make $dy \div dx = 0$ in equation (7), we obtain the cubic $3a^3 - 24ax^2 + 20x^3 = 0$, of which the roots are $+1{\cdot}0689a$, $+0{\cdot}44617a$, $-0{\cdot}31478a$. It is plain that the root we require is $+0{\cdot}44617a$, and this is, therefore, the abscissa of the point of greatest deflection in the extreme segment. Introducing this value for x into the equation (7), we find, for the deflection of the lowest point, the value $13 \,.\, 2\Delta$. It thus appears that the deflection in the extreme segments is more than thirteen times as great as the deflection at the centre.

The point of inflexion in the extreme segment is found by making $d^2y \div dx^2 = 0$ in (7) to be $x = \frac{4}{5}a$.

Deflection in the central segment.—The maximum and minimum deflection are easily seen to occur at the points $x = (\frac{3}{20})^{\frac{1}{2}}\, a$, $x = 0$, $x = -(\frac{3}{20})^{\frac{1}{2}}\, a$, the corresponding deflections being $y = -\frac{4}{5}\Delta$, $y = +\Delta$, $y = -\frac{4}{5}\Delta$. By making $d^2y \div dx^2 = 0$, we find that the points of inflexion in the central segment are identical with the points in which the neutral axis cuts the horizontal axis.

Summary.—Suppose a girder 300 feet long supported in the manner we have been considering. Let Δ be the deflection at the centre O. The neutral axis intersects the horizontal axis at a point 22 feet from O, and it is also remarkable that this is a point of inflexion. At a distance of 38 feet the neutral axis culminates, attaining an elevation of $4\Delta \div 5$ *above the horizontal axis*. From this point the neutral axis approaches the horizontal axis, and crosses it at a point of support 50 feet from O. At this point the neutral axis is inclined to the horizontal axis by an angle equal to $16\Delta \div 100$. The neutral axis now descends rapidly below the horizontal axis. A second point of inflexion is reached at a distance of 70 feet from O, where the deflection is $5{\cdot}6\Delta$; from this the deflection increases to a maximum amount of $13{\cdot}2\Delta$, at a distance of 106 feet from O. This is the lowest point of the neutral axis, which finally ascends to meet the horizontal axis at an angle of $48\Delta \div 100$ in the terminal point of support.

II. *Solution by the* Proposer.

Taking as axes, of x and of y respectively, the horizontal line of the props and the vertical bisector of the rod, and denoting, respectively, by a and b the semi-intervals between the extreme and intermediate pairs of props, by P and Q the equal values of the corresponding pairs of reactions, by W the entire weight of the rod, by E its ordinary modulus of longitudinal elasticity, and by I its horizontal moment of sectional inertia; we have, on the ordinarily received principles, for any point xy on either extreme, and for any point $x'y'$ on the mean segment of its strained axis,

the approximate equations of equilibrium,

$$\mathrm{EI}\,.\,\frac{d^2y}{dx^2} = \mathrm{P}\,(a-x) - \frac{1}{4}\frac{\mathrm{W}}{a}(a-x)^2 \text{ ....(1)},$$

$$\mathrm{EI}\,.\,\frac{d^2y'}{dx'^2} = \mathrm{P}\,(a-x') + \mathrm{Q}\,(b-x') - \frac{1}{4}\frac{\mathrm{W}}{a}(a-x')^2 \text{.........(2)},$$

which, if P and Q were known, would give at once, by integration, the form and all other particulars of the equilibrium of the rod under the circumstances of the case.

By the first integration, remembering that $\frac{dy'}{dx'} = 0$ when $x' = 0$, and $= \frac{dy}{dx}$ when $x' = x = b$, we get at once, from (1) and (2),

$$\mathrm{EI}\,.\,\frac{dy}{dx} = \tfrac{1}{2}\mathrm{P}\,[a^2-(a-x)^2] + \tfrac{1}{2}\mathrm{Q}b^2 - \frac{1}{12}\frac{\mathrm{W}}{a}[a^3-(a-x)^3] \text{ ....(3)},$$

$$\mathrm{EI}\,.\,\frac{dy'}{dx'} = \tfrac{1}{2}\mathrm{P}\,[a^2-(a-x')^2] + \tfrac{1}{2}\mathrm{Q}\,[b^2-(b-x')^2] - \frac{1}{12}\frac{\mathrm{W}}{a}[a^3-(a-x')^3] \text{...(4)},$$

which give the values of the angular deflection at all points of both segments when P and Q are known.

By the second integration, remembering that $y = 0$ when $x = a$, and $= y'$ when $x = x' = b$, we get again, from (3) and (4),

$$\mathrm{EI}\,.\,y = -\tfrac{1}{2}\,(\mathrm{P}a^2 + \mathrm{Q}b^2 - \tfrac{1}{3}\mathrm{W}a^2)\,(a-x) + \tfrac{1}{6}\mathrm{P}\,(a-x)^3 - \frac{1}{48}\frac{\mathrm{W}}{a}(a-x)^4 \text{......(5)},$$

$$\mathrm{EI}\,.\,y' = -\tfrac{1}{2}\,(\mathrm{P}a^2 + \mathrm{Q}b^2 - \tfrac{1}{3}\mathrm{W}a^2)\,(a-x')$$
$$+ \tfrac{1}{6}\mathrm{P}\,(a-x')^3 + \tfrac{1}{6}\mathrm{Q}\,(b-x')^3 - \frac{1}{48}\frac{\mathrm{W}}{a}(a-x')^4 \text{..................(6)},$$

which give the values of the linear deflection at all points of both segments when P and Q are known.

To determine the values of P and Q, and thus complete the solution of the question. Since y or $y' = 0$ when x or $x' = b$, we get at once, from (5) or (6), the linear relation between them

$$(\mathrm{P}a^2 + \mathrm{Q}b^2 - \tfrac{1}{3}\mathrm{W}a^2) - \tfrac{1}{3}\mathrm{P}\,(a-b)^2 + \frac{1}{24}\frac{\mathrm{W}}{a}(a-b)^3 = 0 \text{............(7)},$$

which, combined with the evident relation $\mathrm{P}+\mathrm{Q} = \tfrac{1}{2}\mathrm{W}$, gives immediately for P and Q the general values

$$\mathrm{P}\,(a-b)\,[(a+b) - \tfrac{1}{3}\,(a-b)] = \frac{1}{6}\frac{\mathrm{W}}{a}[a^3 - 3ab^2 - \tfrac{1}{4}\,(a-b)^3] \text{............(8)},$$

$$\mathrm{Q}\,(a-b)\,[(a+b) - \tfrac{1}{3}\,(a-b)] = \frac{1}{6}\frac{\mathrm{W}}{a}[a^3 + 2a^2b - ab^2 + \tfrac{1}{4}\,(a-b)^3] \text{... (9)};$$

whatever be the ratio of the semi-intervals a and b between the extreme and intermediate pairs of props, provided, of course, that the intervals themselves, as all along supposed, are concentric.

When $a = 3b$, as supposed in the question, the general relations (8) and (9) give for P and Q the particular values

$$\mathrm{P} = \tfrac{4}{30}\mathrm{W}, \text{ and } \mathrm{Q} = \tfrac{11}{30}\mathrm{W} \text{........................(10)},$$

which, accordingly, is the required distribution of the weight of the rod between the two pairs of props.

To find the points of inflection, if any, of the extreme segments of the

rod. Since for them $\frac{d^2y}{dx^2} = 0$, therefore, at once, from (1) and (10),

$$(a-x)^2 - \tfrac{8}{15}a\,(a-x) = 0 \quad\text{..............................(11)},$$

which, rejecting the solution $x = a$, corresponding to the obvious inflections at the terminal props, gives $x = \frac{7}{15}a$ for the distance of the other point of inflection of either extreme segment from the centre of the rod.

To find the points of inflection, if any, of the mean segment of the rod. Since for them $\frac{d^2y'}{dx'^2} = 0$, therefore, at once, from (2) and (10),

$$(a-x')^2 - 2a\,(a-x') + \tfrac{44}{45}a^2 = 0 \quad\text{........................(12)},$$

which gives $x' = \pm\frac{1}{3}\frac{a}{\sqrt{5}}$ for the equal distances, in opposite directions, of the two points of inflection of the mean segment from the centre of the rod.

To find the points of greatest linear deflection of the mean segment of the rod. Since for them $\frac{dy'}{dx'} = 0$, therefore, from (4) and (10), after a few trifling reductions, $\quad x'^3 - \frac{1}{15}a^2x' = 0$ ....(13), which, rejecting the solution $x' = 0$, corresponding to the obvious horizontality of the rod at its centre, gives $x' = \pm\frac{a}{\sqrt{15}}$ for the equal distances, in opposite directions, of the two points of greatest linear deflection of the mean segment from the centre of the rod.

The two extreme segments also have, of course, points of greatest linear deflection equidistant in opposite directions from the centre of the rod; but the cubic obtained from (3) and (10) for the determination of their distances from the centre, viz.,

$$(a-x)^3 - \tfrac{4}{5}a\,(a-x)^2 + \tfrac{2}{45}a^3 = 0 \quad\text{........................(14)},$$

hough having all its roots real, and one of them lying within the limits of the segments, admits only of an approximate solution in this case.

5783. (By S. Tebay, B.A.) — If m cards, consisting of s suits of $a_1, a_2, \ldots\ldots a_s$ cards respectively, be shuffled, find the probability that each suit consists of p partitions.

I. *Solution by the* Rev. W. A. Whitworth, M.A.

The first suit of a cards can be arranged into p groups in

$$\frac{a_1!\; a_1-1!}{a_1-p!\; p!\; p-1!}$$

ways. (*Choice and Chance*, Prop. xvii.) And so for the other suits. Hence all the suits can be arranged, into p groups each, in

$$\frac{(a_1!\, a_2! \ldots a_s!)\,.\,(a_1-1!\, a_2-1! \ldots a_s-1!)}{a_1-p!\; a_2-p!\; a_3-p! \ldots a_s-p!\; (p!\, p-1!)^s}$$

ways ($\equiv$ N ways, suppose). We have now to arrange our ps groups

(which are of s sorts, p of every sort) so that no two groups of the same sort may come together. If we take only two of the sorts, we can arrange the $2p$ things in $2\,.\,p\,!\,p\,!$ ways. A third set of p groups can be introduced among these in $2p+1\,!\div p+1\,!$ ways. A fourth set can then be introduced in $3p+1\,!\div 2p+1\,!$ ways, and so on. Finally the s^{th} set can be introduced in $sp-p+1\,!\div sp-2p+1\,!$ ways. Hence all the ps groups can be arranged in $2\,.\,p!\,sp-p+1\,!\div(p+1)$ ways. Therefore the whole number of favourable arrangements of the m cards is $2\text{N}\,.\,p\,!\,sp-p+1\,!\div(p+1)$, and the required chance will be the ratio of this number to $m\,!$ the total number of possible arrangements. Hence the chance is

$$\frac{2\,.\,sp-p+1\,!\,.\,(a_1\,!\,a_2\,!\,\ldots\,a_s!)\,.\,(a_1-1\,!\,a_2-1\,!\,\ldots\,a_s-1\,!)}{(p+1)\,(p\,!)^{s-1}\,(p-1\,!)^s\,a_1-p\,!\,a_2-p\,!\,\ldots\,a_s-p\,!\,m\,!}.$$

II. *Solution by the* Proposer.

Let the equations $\alpha_1+\alpha_2+\ldots+\alpha_p=a_1$, $\beta_1+\beta_2+\ldots+\beta_p=a_2$, $\gamma_1+\gamma_2+\ldots+\gamma_p=a_3$, &c., admit of α, β, γ, &c. solutions, and let one form of partition be represented by

$$\alpha_1\,\beta_1\,\gamma_1\ldots\alpha_2\,\beta_2\,\gamma_2\ldots\alpha_p\,\beta_p\,\gamma_p\ldots.$$

This arrangement gives $(s\,!)^p$ variations, including repetitions. To find the number of repetitions, let letters of the same name be joined throughout the p groups. While these are unchanged, the other letters give $(s-1\,!)^2$ $(s-2\,!)^{p-2}$ variations. Now, there are $p-1$ connections, and any two consecutive groups can form s connections, while the others run out in consecutive pairs to both extremes, giving $s-1$ connections at each step, or $(p-1)\,s\,(s-1)^{p-2}$ in all; consequently the number of repetitions

$$=(p-1)\,s\,(s-1)^{p-2}\,(s-1\,!)^2\,(s-2\,!)^{p-2}=(p-1)\,s\,(s-1\,!)^p.$$

Hence the number of partitions

$$=(s\,!)^p-(p-1)\,s\,(s-1\,)^p;$$

and, interchanging the limits, the chance required

$$=(p\,!)^s\left\{(s\,!)^p-(p-1)\,s\,(s-1\,!)^p\right\}\alpha\beta\gamma\ldots\div\frac{m\,!}{a_1\,!\,a_2\,!\,\ldots\,a_s\,!}.$$

Example. Let $p=2$, $s=3$, and take $a_1=3$, $a_2=4$, $a_3=5$; then $\alpha=1$, $\beta=2$, $\gamma=2$. Also $m=12$; and the chance $=$

$$(2\,!)^3\left\{(3\,!)^2-3\,.\,(2\,!)^2\right\}.\,4\div\frac{12\,!}{3\,!\;4\,!\;5\,!}=\frac{16}{1155}.$$

[In regard to the foregoing solution, Mr. Whitworth inquires: "Why does not Mr. Tebay evaluate $\alpha\beta\gamma\ldots$ in his final result? We could they test the accuracy of his result by applying it to particular cases. But, if I understand the question aright, Mr. Tebay's reasoning is quite unsound, and his result wrong. Consider the case when $p=1$, then the question is: to find the chance that all the cards of each suit are together.

My result comes to $\dfrac{s\,!\,a_1\,!\,a_2\,!\,\ldots\,a_s!}{m\,!}$, which is obviously right.

Mr. Tebay's comes to $\dfrac{s-1\,!\,a_1\,!\,a_2\,!\,\ldots\,a_s!}{m\,!}$, which is obviously wrong."

To this Mr. Tebay rejoins that, after a careful examination, he is con-

vinced that his solution is right, and that Mr. WHITWORTH and he only differ in the fact that he has partitioned *numbers* and Mr. WHITWORTH partitioned *things;* that, in short, Mr. WHITWORTH seems to consider each suit made up of different cards, as if marked 1, 2, 3, &c., while he has himself considered them as of one colour; namely, a_1 red, a_2 white, &c. Mr. TEBAY adds that there is a difference between the partitioning of things and the partitioning of numbers; and, for this reason, he considers that both solutions may stand as, from different points of view, equally correct.

Mr. WHITWORTH, having seen Mr. TEBAY's rejoinder, still adheres to his former assertion, and points out that the chance of the a_1 red cards being in p partitions after shuffling cannot be affected by the question whether these cards are *inter se* distinguishable or indistinguishable.]

5875. (By the Rev. Dr. HAUGHTON, F.R.S.)—Show that, within the limits of NEWTON's construction for the disturbing force, the *equilibrium tidal surface* (statical problem) of an equatorial canal of constant width and depth (without rotation), attracted by the moon always in the equator, is

$$\log(\rho) = \alpha \cos 2\phi + \beta.$$

Solution by J. J. WALKER, M.A.

In this case I resolve the forces acting on the particle of water P along PT (ρ), T being the Earth's centre, and PY parallel to the line joining centres of Earth and Moon. The disturbing forces in these directions are $\dfrac{k\rho}{R}$ and $\dfrac{3k\rho\cos\phi}{R}$ respectively, within the specified limits; and adding to the former $\dfrac{kg'a^2}{R\rho^2}$ for Earth's attraction, where $g' = \dfrac{gR}{k}$, the condition that the forces on P shall have a resultant along PN the normal to the free surface, meeting the line of centres in N, is

$$\rho + \frac{g'a^2}{\rho^2} : 3\rho\cos\phi = \text{PT} : \text{TN} = d(\rho\cos\phi) : d\rho,$$ [see Quest. 5964,]

or $$\left(\rho + \frac{g'a^2}{\rho^2}\right) d\rho = 3\rho\cos\phi \, d(\rho\cos\phi),$$

whence $$\rho^2 - \frac{2g'a^2}{\rho} + C = 3\rho^2\cos^2\phi.$$

If, in the expressions for the disturbing forces and Earth's attraction, we take $\rho = a$, and $d(\rho\cos\phi) = \rho d\cos\phi$, then the equation to be integrated

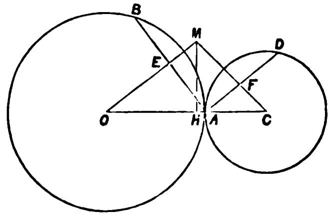
will be $$\frac{a+g'}{a\rho}\, d\rho = 3\cos\phi \, d(\cos\phi),$$

whence $$\frac{a+g'}{a}\log\rho = \tfrac{3}{2}\cos^2\phi + C,$$

which is of the form proposed in the Question, but the one above is obviously more exact.

5855. (By Professor DARBOUX.)—Par le point de contact A de deux circonférences données, on mène deux cordes AB et AD qui soient dans un rapport donné ; et des centres O, C on abaisse des perpendiculaires sur ces cordes. On demande le lieu géométrique du point de rencontre M de ces perpendiculaires.

Solution by Professor COCHEZ ; G. TURRIFF, M.A. ; *and others.*



Du point M abaissons la perpendiculaire MH sur la ligne des centres. Les triangles semblables MOH et AOE, MCH et AFC, donnent

$$\frac{MO}{MH} = \frac{OA}{AE}, \quad \frac{CM}{MH} = \frac{CA}{FA}.$$

Divisant membre à membre ces deux relations,

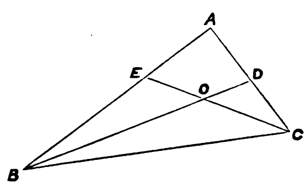
$$\frac{OM}{CM} = \frac{OA \cdot FA}{CA \cdot EA} = \text{Constante};$$

puis que $\dfrac{FA}{EA}$ est constant par hypothèse.

Le lieu est donc une circonférence décrite sur la droite qui joint les points conjugués harmoniques partageant OC dans le rapport donné.

5866. (By H. N. CAPEL, LL.B)—Given the two lines bisecting the acute angles of a right-angled triangle ; determine the triangle.

Solution by MORGAN BRIERLEY ; EDWYN ANTHONY, M.A. ; *and others.*



From any point O, draw OB equal to one of the given bisectors ; make $\angle BOC = \frac{3}{4}\pi$; take OC = the other given bisector ; join BC, and make $\angle ABO = OBC$; and $\angle OCA = OCB$; then ABC is the required triangle. For, from the triangle BOC, we have

$$\frac{3\pi}{4} + \tfrac{1}{2}(B+C) = \pi ;$$

therefore $\qquad B + C = \frac{1}{2}\pi, \quad$ therefore $A = \frac{1}{2}\pi$.

[Our correspondents consider the bisectors in the question to terminate at their point O of intersection. These bisectors may, however, with equal propriety, be taken as terminated at the points D, E in the opposite sides of the triangle ; and in this form the problem has been reproposed for solution as Question 5885.]

5853. (By Professor CROFTON, F.R.S.)—Explain the following paradox:—One bag (A) contains 5 cheques, whereof 2 are known to be over £5 and 3 under. Another bag (B) contains 7 cheques, 4 over £5 and 3 under. Find the chance, if a cheque be drawn from A, that it exceeds in amount one drawn from B. There are 35 cases, all equally probable; 6 evidently favourable, 12 unfavourable, and 17 doubtful. In these 17 cases, as all we know is that the 2 cheques drawn are both over £5, or both under, it is an even chance that one exceeds the other. Hence the chance required, that the cheque from A exceeds that from B, is $\frac{6}{35}+\frac{1}{2}\left(\frac{17}{35}\right)=\frac{29}{70}$. But the chance that the cheque from A exceeds £5 is $\frac{2}{5}=\frac{28}{70}$, so that there is a *less* chance that the cheque drawn from A exceeds £5 than that it exceeds an (unknown) quantity which is likely to be greater than £5; the odds being 4 to 3. *A priori*, one would say it must be the other way.

Solution by W. J. MACDONALD, M.A.

Though the chance that a cheque from A exceeds one from B is $\frac{29}{70}$, yet the chance that the latter exceeds £5 is only $\frac{4}{7}$; therefore the chance that a cheque from A exceeds a quantity greater than £5 is only $\frac{29}{70}\times\frac{4}{7}=\frac{58}{245}$.

5831. (By Professor MINCHIN, M.A.)—If S and E be a conic and its evolute, and if N be the conic whose intersections with S are the feet of the normals drawn to S from a point P: prove that (1) the perpendicular from P on its polar with respect to S is a tangent to N at P; (2) the centre of gravity of the feet of the four normals is the middle point of the line joining the centres of S and N; (3) the locus of points whence normals drawn to S form a harmonic pencil is $E+27a^2b^2c^4x^2y^2=0$; (4) E being $\equiv C^3+27a^2b^2c^4x^2y^2$, if P be any point on the conic C, the locus of the centre of N is S.

Solution by R. GRAHAM, M.A.; S. JOHNSTON, M.A.; *and others.*

1. Let $\quad S\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0,\quad N\equiv c^2xy+b^2y'x-a^2x'y=0,$

$$E\equiv(a^2x^2+b^2y^2-c^4)^3+27a^2b^2c^4x^2y^2=C^3+27a^2b^2c^4x^2y^2=0.$$

The conic N manifestly passes through $x'y'$, the common point of the four normals. The equation of the tangent to N at $x'y'$ is

$$\frac{xy'}{b^2}-\frac{yx'}{a^2}=\frac{c^2x'y'}{a^2b^2},\ \text{which is at right angles to}\ \frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

the polar of $x'y'$ with respect to S.

2. If ϕ be the eccentric angle of the foot of one of the normals through $x'y'$, then
$$\frac{ax'}{\cos\phi}-\frac{by'}{\sin\phi}=c^2,$$
which can be written in either of the forms

$$c^4\cos^4\phi-2ac^2x'\cos^3\phi+C'\cos^2\phi+2ac^2x'\cos\phi-a^2x'^2=0\ \ldots\ldots(1),$$

or $$c^4\sin^4\phi+2bc^2y'\sin^3\phi+C'\sin^3\phi-2bc^2y'\sin\phi-b^2y'^2=0.$$

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity of the feet of the four normals through (x', y'), we have from these last equations

$$\bar{x} = \frac{a^2x'}{2c^2} \text{ and } \bar{y} = -\frac{b^2y'}{2c^2}.$$

The coordinates of the centre of N are $x = \frac{a^2x'}{c^2}$, $y = -\frac{b^2y'}{c^2}$;

hence the coordinates of the middle point of the line joining the centres of S and N are equal to $\bar{x}$ and $\bar{y}$ respectively.

3. The intercept on the axis of x by one of the normals is $\frac{c^2}{a}\cos\phi$; if we multiply each root of equation (1) by $\frac{c^2}{a}$, we shall have an equation whose roots are the intercepts made by the four normals through (xy); viz.,

$$a^2\lambda^4 - 2a^2x\lambda^3 + C\lambda^2 + 2c^2x\lambda - c^4x = 0.$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the roots of this biquadratic; then, since the normals form a harmonic pencil, $\frac{\lambda_3-\lambda_4}{\lambda_2-\lambda_3} = \frac{\lambda_1-\lambda_4}{\lambda_1-\lambda_2}$,

or $\frac{1}{2}(\lambda_1\lambda_3+\lambda_2\lambda_4) = \frac{C}{6a^2}$ = a root of SIMPSON's reducing cubic;

hence we find $\frac{C^3}{27a^6} + \frac{2c^4b^2x^2y^2}{a^4} = 0$ or $E + 27a^2b^2c^4x^2y^2 = 0$.

4. By Art. 2, $ax' = c^2\frac{x}{a}$, and $by' = -c^2\frac{y}{b}$; where x, y are the coordinates of the centre of N. Substituting in $C=0$, since $x'y'$, by hypothesis, lies on C, we find $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ for the locus of the centre of N.

5906. (By Professor MATZ, M.A.)—In a given ellipse, determine the average length of all the radius-vectors that can be drawn from one of the foci through the arc intercepted by the focal ordinates.

Solution by R. E. RILEY, B.A.; A. W. SCOTT, M.A.; *and others.*

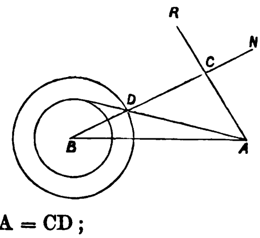
The polar equation of the ellipse being $\rho = \frac{a(1-e^2)}{1-e\cos\theta}$, the average required is, putting $\theta_1 = \cos^{-1}\left(\frac{2e}{1+e^2}\right)$,

$$\frac{\int\rho d\theta}{\int d\theta} = \frac{a(1-e^2)}{\frac{1}{2}\pi-\theta_1}\int_{\theta_1}^{\frac{1}{2}\pi}\frac{d\theta}{1-e\cos\theta}$$

$$= \frac{2a(1-e^2)^{\frac{1}{2}}}{\frac{1}{2}\pi-\theta_1}\left\{\tan^{-1}\left[\tan\tfrac{1}{2}\theta\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\right]\right\}_{\theta_1}^{\frac{1}{2}\pi} = \frac{b\sin^{-1}e}{\tan^{-1}e}.$$

5798. (By Professor COCHEZ.)—Construire un triangle rectangle, connaissant les sommes de chacun des côtés avec la perpendiculaire abaissée du sommet de l'angle droit sur l'hypotenuse.

Solution by R. E. RILEY, B.A.; J. O. REGAN; *and others.*



We are given $a+p$, $b+p$, and the hypotenuse AB. From centre B, and with radius $a-b$, draw a circle; and from the centre B, with radius whose square is $\frac{1}{2}(a-b)$, draw another circle. From A draw a tangent to the latter circle, cutting the former in D. Join BD and produce it to N, making BN $= a+p$. Draw AC at right angles to BN, and produce it to R, making CR = CN. Then, if AR $= b+p$, ABC is the triangle required. For $\angle$ CDA = 45°, therefore CA = CD;

DN $= a+p-(a-b) = b+p$; and DN = AR, therefore AR $= b+p$; and ABC is the required triangle.

5896. (By E. W. SYMONS.)—Prove that the volume of a tetrahedron, whose vertices are any vertex of a given tetrahedron, and the centroids of the faces meeting in that vertex, is $\frac{2}{27}$ of the volume of the given tetrahedron.

Solution by Rev. D. THOMAS, M.A.; R. TUCKER, M.A.; *and others.*

Let O be a vertex, ABC the opposite face, and D, E, E the middle points of BC, CA, AB. Then the volume of the tetrahedron in the question is clearly $\frac{8}{27}$ ODEF; and

$$\text{vol. ODEF} : \text{vol. OABC} = \triangle\text{DEF} : \triangle\text{ABC} = 1 : 4,$$

therefore given volume $= \frac{1}{4}$ of $\frac{8}{27}$ of OABC $= \frac{2}{27}$OABC.

5862. (By EDWYN ANTHONY, M.A.)—Prove that $n^{13}-n$ is always divisible by 2730, whatever whole number n may be.

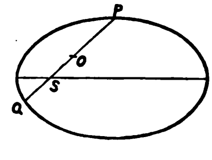
Solution by R. TUCKER, M.A.; W. J. MACDONALD, M.A.; *and others.*

$$\text{N} \equiv n^{13}-n = 2730 = n(n^{12}-1).$$

By Fermat's theorem, since $n^{12}-1 = (n^6-1)(n^6+1)$, $n^{12}-1$ is divisible by 13 and by 7; also, since $n^{12}-1 = (n^4-1)(n^8+n^2+1)$, it is divisible by 5; and, as it is divisible by $(n-1)n(n+1)$, it is also divisible by 6; therefore N is divisible by 13 . 7 . 5 . 6 – 2730.

5887. (By EDWYN ANTHONY, M.A.)—If O be the middle point of a focal chord PSQ of an ellipse, and SP equals its semi-major-axis; show that the ratio of OS to OP equals the square of the eccentricity.

Solution by C. F. D'ARCY, M.A.; A. W. SCOTT, M.A.; *and others.*



The point P will be at an extremity of the minor axis. If L be semi-latus-rectum, we have

$$\frac{1}{SP}+\frac{1}{SQ}=\frac{2}{L};$$

hence $\quad SQ=\dfrac{b^2}{a(1+e^2)},\quad$ and $\quad PQ=\dfrac{2a}{1+e^2};$

therefore $\quad OS=SP-\tfrac{1}{2}PQ=\dfrac{ae^2}{1+e^2}=e^2.OP.$

5828. (By Professor DARBOUX.)—On coupe une pyramide triangulaire SABC par un plan parallèle à la base; ce plan rencontre les arètes latérales SA, SB, SC en A′, B′, C′; on mène ensuite les plans CA′B′, AB′C′, BC′A′. Soit P leur point commun. Déterminer le lieu décrit par le point P lorsque le plan A′BC′ se déplace en demeurant parallèle à la base.

Solution by R. GRAHAM, M.A.; T. R. TERRY, B.A.; *and others.*

Let BC′ and B′C intersect in O_2 then SO produced will evidently bisect BC. A′O is the line of intersection of the planes BC′A′ and CA′B′, which lies in the plane through AS bisecting the opposite edge BC. Hence the point whose locus we seek lies on the line joining S with the centre of gravity of the face ABC.

5870. (By J. L. MACKENZIE, B.A.)—The evolute of a non-singular cubic has in general twenty-seven cusps. If P be a point on the cubic corresponding to any of these cusps, shew that the remaining points in which the circle of curvature at P cuts the cubic are collinear with the *second tangential* of P. [The term "second tangential" is employed by SALMON in his *Higher Plane Curves*, 2nd ed., Art. 155. The tangent at any point P on a cubic meets the curve again in a point which is called the *tangential* of P; and the tangent at the latter point meets the curve again in a point which is the *second tangential* of P.]

Solution by the PROPOSER.

When there is a cusp on the evolute of the cubic, three consecutive normals to the cubic pass through a point, and therefore four consecutive points on the cubic lie on the circle of curvature at P. This circle cuts the cubic again in two points A, B; and the straight line AB cuts the cubic again in R, the point-residual of the four consecutive points at P. Again, the tangent at P, taken twice, passes through the four consecutive points at P, and cuts the cubic again in two consecutive points at Q, the first tangential of P; and the tangent at Q cuts the cubic again in R′, the second tangential of P. And, by the theory of residuation, this point R′ is the same as R, the point-residual of the four consecutive points at P.

5802. (By T. R. TERRY, M.A.)—If O be a fixed point in a rigid body, and the direction cosines of the lines OA, OB, OC be represented by (l, m, n), (l', m', n'), (λ, μ, ν), respectively; show that a rotation of the body round the axis OA through an angle 2θ, followed by a rotation round the axis OB through an $\angle 2\theta'$, is equivalent to a rotation round the axis OC through an angle 2ϕ, where

$$\frac{\cos\phi}{\cos\theta'\cos\theta-\sin\theta'\sin\theta\,(ll'+mm'+nn')}=\frac{\lambda\sin\phi}{X}=\frac{\mu\sin\phi}{Y}=\frac{\nu\sin\phi}{Z},$$

where

$$X = l\cos\theta'\sin\theta+l'\cos\theta\sin\theta'+\sin\theta'\sin\theta\,(m'n-mn'),\ Y = \&c.,\ Z = \&c.$$

Solution by J. J. WALKER, M.A.

These expressions may be found substantially in Hamilton's earlier papers on Quaternions. The first application of his new method, which he communicated to the R. I. A. (*Proc.*, Vol. III., 1834), was to the composition of translations and rotations; and he gave for the rotation of a vector β round a fixed unit vector α, through an angle 2θ, the beautiful formula, $\beta' = (\cos\theta+\alpha\sin\theta)\,\beta\,(\cos\theta-\alpha\sin\theta)$. Similarly, if by a second rotation of β' about α' through an $\angle 2\theta'$, it becomes β'', then

$$\beta'' = (\cos\theta'+\alpha'\sin\theta')\,\beta'\,(\cos\theta'-\alpha'\sin\theta')$$

$$= (\cos\theta'+\alpha'\sin\theta')(\cos\theta+\alpha\sin\theta)\,\beta\,(\cos\theta-\alpha\sin\theta)(\cos\theta'-\alpha'\sin\theta').$$

By a single rotation through an angle 2ϕ, round an axis α_1, β may be made to coincide with β'' if $\cos\phi$ is identified with the scalar part of the product $(\cos\theta'+\alpha'\sin\theta')(\cos\theta+\alpha\sin\theta)$, which by actual multiplication is $\cos\theta\cos\theta'-\sin\theta\sin\theta'\,(ll'+mm'+nn')$, since, i, j, k being unit vectors in the directions of the axes of coordinates,

$$\alpha = li+mj+nk,\quad \alpha' = l'i+m'j+n'k;$$

while, by the identification of the vector part of $\cos\phi+\alpha_1\sin\phi$, viz., $(\lambda i+\mu j+\nu k)\sin\phi$, with that of the same product,

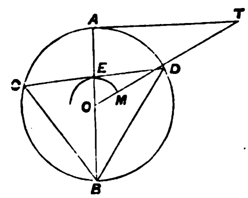
$$\lambda\sin\phi = X,\quad \mu\sin\phi = Y,\quad \nu\sin\phi = Z.$$

Thus the fractions, proposed to be proved equal, are actually each equal to unity. From the formulæ thus found, Hamilton deduced his construction of the resultant axis as the radius drawn to the third corner of a

spherical triangle, the other two corners of which are determined by the two component axes, the magnitudes of the angles at these corners being θ, θ' respectively; while ϕ is equal to the supplement of the angle at the first. [See also TAIT's "*Quaternions,*" 2nd ed., Arts. 351, 352.]

5768. (By Professor DUPAIN.)—Déterminer sur un diamètre AB d'une sphère de rayon R, un point tel que si l'on mène par ce point un plan perpendiculaire à ce diamètre, la surface de la zône sphérique limitée par le plan et contenant le point A, soit équivalente à la surface latérale du cône qui a pour base le cercle d'intersection de la sphère et du plan et pour sommet le point B. Celà étant, calculer le rapport du volume de ce cône au volume de la sphère.

Solution by Professor COCHEZ; H. L. ORCHARD, B.A.; *and others.*



Soit E le point cherché. Posons OE $= x$, on doit avoir

$$2\pi R \,.\, AE = \pi\, CE \,.\, BC,$$

ou $\quad 2R \,.\, AE = CE \,.\, BC \ldots\ldots\ldots\ldots(1),$

R étant le rayon de la sphère. Mais

$$AE = R-x, \quad CE = (R^2-x^2)^{\frac{1}{2}},$$
$$CB = [2R\,(R+x)]^{\frac{1}{2}}.$$

Portant ces valeurs dans (1), on

$$[2R\,(R-x)]^{\frac{1}{2}} \left\{[2R\,(R-x)]^{\frac{1}{2}}-(R+x)\right\} = 0,$$

égalité satisfaite pour $x = R$; dans ce cas les deux surfaces sont nulles, et

pour $\quad [2R\,(R-x)]^{\frac{1}{2}} = R+x,$ qui donne $x^2+4Rx-R^2 = 0,$

d'où $\quad x' = R\,(\sqrt{y}-z), \; x'' = -R\,(z+\sqrt{y});$

cette seconde valeur x'' ne saurait convenir puisque sa valeur absolue est plus grande que R. Il reste donc comme solutions

$$x = 0, \quad x = R(\sqrt{y}-z).$$

Le rapport K des volumes est $K = \dfrac{\frac{1}{3}\pi\,(R^2-x^2)\,(R+x)}{\frac{4}{3}\pi R^3} = \dfrac{(R^2-x^2)\,(R+x)}{4R^3}$

ou, en remplaçant x par sa valeur, $K = 7-3\sqrt{y}.$

La valeur de x se construit facilement. Pour cela on mène en A une tangente $AT = 2R$; on joint OT, on a $OT = R\sqrt{y}$, on prend $TM = 2R$ et on décrit de O comme centre avec OM pour rayon un cercle qui coupe OA en E, E est le point qui répond à la question.

5851. (By Prof. TOWNSEND, F.R.S.)—A material particle being supposed in equilibrium, in a free space, under the action, attractive or repul-

sive, of a system of masses distributed in any manner in the space; show, generally, that its equilibrium, when not absolutely neutral, is always perfectly stable for attractive, and perfectly unstable for repulsive, action, or is, on the contrary, always intermediate, according as the law of the action is a direct or an inverse power of the distance.

I. *Solution by* W. J. CURRAN SHARP, M.A.

If m_1, m_2, m_3. &c. be the masses; a, b, c, &c. their coordinates; $r_1\, r_2$, &c. the distances from (xyz), the attracted particle; so that

$$r_1^2 = (a_1-x)^2+(b_1-y)^2+(c_1-z)^2;$$

the attractions parallel to the axes are for a force varying as the n^{th} power of

$$\begin{aligned} &\mp\Sigma\, r_1^{n-1}(a_1-x) = 0 \equiv \text{X parallel to axis of } x,\\ &\mp\Sigma\, r_1^{n-1}(b_1-y) = 0 \equiv \text{Y} \quad ,, \quad y,\\ &\mp\Sigma\, r_1^{n-1}(c_1-z) = 0 \equiv \text{Z} \quad ,, \quad z,\end{aligned}$$

the upper sign corresponding to an attractive and the lower to a repulsive force.

Then the equilibrium will be stable, unstable, or intermediate, according as $\int(\text{X}dx+\text{Y}dy+\text{Z}dz)$ is a maximum, a minimum, or neither; and this integral becomes $\pm\dfrac{\Sigma r_1^n}{n}$, and using the notation of WILLIAMSON'S *Diff. Calc.*, p. 183,

$$\begin{aligned}\text{A} &= \pm\,\Sigma\left\{r_1^{n-1}+(n-1)\,r_1^{n-2}(a_1-x)^2, \&c.\right\},\\ \text{H} &= \pm\,\Sigma\,(n-1)\,r_1^{n-2}(a_1-x)(b_1-y),\ \&c.;\end{aligned}$$

and the expression $\text{A}\alpha^2+$ &c., reduces to

$$\pm\,\Sigma r_1^{n-1}(\alpha^2+\beta^2+\gamma^2)\pm(n-1)\,\Sigma r_1^{n-2}\left\{\overline{a_1-x}\,\alpha+\overline{b_1-y}\,\beta+\overline{c_1-z}\,\gamma\right\}^2,$$

which is essentially positive for an attractive and negative for a repulsive force if n be positive, and is intermediate for negative values of n.

II. *Solution by the* PROPOSER.

To prove this, it is only necessary to show, in accordance with a well-known general principle in dynamics, the application of which to the case in question is obvious, that, special circumstances excepted, the potential V of the system is always a minimum in the former case, and only sta-[illegible]nary in the latter case; which may be readily done as follows:—

[illegible]onceiving the particle to receive a small displacement $\xi\eta\zeta$ from its [illegible]tion of equilibrium xyz under the action of the system; it is to be [illegible]n that the function, representing accurately in sign and approximately [illegible]nitude the consequent change in V; viz.,

$$[illegible]\frac{V}{[illegible]}\xi^2+\frac{d^2V}{dy^2}\eta^2+\frac{d^2V}{dz^2}\zeta^2+2\frac{d^2V}{dydz}\eta\zeta+2\frac{d^2V}{dzdx}\zeta\xi+2\frac{d^2V}{dxdy}\xi\eta\Big)$$

[illegible]ommon sign and is positive for all values of $\xi\eta\zeta$ in the former [illegible]s not in the latter case.

[illegible]ally, for any power n of the distance,

$$\Sigma\,(m\,.\,r^{n+1}) = \Sigma\left[m\left\{(x-a)^2+(y-b)^2+(z-c)^2\right\}^{\frac12(n+1)}\right]$$

where abc, $a'b'c'$, &c. are the coordinates of m, m', &c. in the space; therefore, performing the several differentiations, and adding together and arranging the several results, we have, for the function in question, the value

$$\Sigma\left[mr^{n-3}\left\{[(x-a)^2+(y-b)^2+(z-c)^2]\,[\xi^2+\eta^2+\zeta^2] + [n-1]\,[(x-a)\,\xi+(y-b)\,\eta+(z-c)\,\zeta]^2\right\}\right];$$

or, as it may be written more shortly

$$\Sigma\left[mr^{n-3}\left\{r^2\rho^2+(n-1)\,r^2\rho^2\cos^2\theta\right\}\right],$$

where ρ is the vector of the displacement, and θ the angle between the directions of r and ρ.

But this manifestly preserves a common sign, and is positive, for all values of ρ and θ, when n is positive, and does not preserve a common sign at all, as ρ and θ vary, when n is negative; and therefore, &c., as regards the property.

The quadric cone, real or imaginary, whose equation, in rectangular coordinates to the position of equilibrium xyz as origin, is

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0,$$

where $$A=\Sigma\left[mr^{n-3}\left\{r^2+(n-1)\,(x-a)^2\right\}\right],\ \&c.,$$
$$F=\Sigma\left[mr^{n-3}\left\{(n-1)\,(y-b)\,(z-c)\right\}\right],\ \&c.,$$

evidently divides, in all cases, the space about the point into the two different regions, for which the changes in V resulting from the displacement of the particle have opposite signs in the above.

The vector ρ of the displacement being supposed constant; the squared reciprocal of any radius of the central quadric to the same origin and axes,

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=k^2,$$

where k is any constant, represents as evidently, in all cases, in magnitude and sign, the change in V resulting from a displacement of the particle in the direction of the radius.

2495. (By W. S. B. Woolhouse, F.R.A.S.) — Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude, such that, if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum.

Solution by the Proposer.

This problem was originally proposed by me as an exercise upon the practical application of the very general theorem established in Question 2471 (*Reprint*, Vol. VIII., page 100). It will here be convenient in the first place to reproduce the enunciation of the theorem, which is as follows :—

Theorem.—Let a given surface having a convex boundary of *any form whatever*, be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; then, if h, k denote the radii of gyration round the axes, and M the total area, we shall have

$$h^2 = \int x^2\,dx\,\frac{y+y'}{\mathrm{M}}, \quad k^2 = \tfrac{1}{3}\int dx\,\frac{y^3+y'^3}{\mathrm{M}}.$$

Let also
$$\mathrm{A} = \int x^2\,dx\,\frac{y^3+y'^3}{\mathrm{M}}, \quad \mathrm{B} = \int x\,dx\,\frac{y\int y^2dx + y'\int y'^2\,dx}{\mathrm{M}},$$
$$\mathrm{C} = \tfrac{1}{3}\mathrm{A} + 3\mathrm{B};\ \text{then—}$$

1. If *three* points be taken at random, on the given surface, the average area of the triangle connecting them, in parts of the total area, is $(\Delta) = \dfrac{h^2k^2+\mathrm{C}}{\mathrm{M}^2}$; and the average square of the area, expressed in parts of the square of the total area, is $(\Delta^2) = \frac{3}{2}\cdot\dfrac{h^2k^2}{\mathrm{M}^2}$.

2. If *four* points be taken at random, the probability that the quadrilateral shall be reentrant $= 4\,(\Delta)$.

3. If *five* points be taken at random on the surface, the probability of

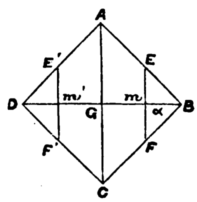
$$\begin{aligned} &\text{a convex pentagon} &&= 1 - 10\,(\Delta) + 10\,(\Delta^2),\\ &\text{one reentrant point} &&= 10\,(\Delta) - 20\,(\Delta^2),\\ &\text{two reentrant points} &&= 10\,(\Delta^2). \end{aligned}$$

Note.—When the axis of x is diametral, then

$$h^2 = \frac{2}{\mathrm{M}}\int yx^2\,dx, \quad k^2 = \frac{2}{3\mathrm{M}}\int y^3\,dx,$$
$$\mathrm{A} = \frac{2}{\mathrm{M}}\int y^3x^2\,dx, \quad \mathrm{B} = \frac{2}{\mathrm{M}}\int yx\,dx\int y^2\,dx.$$

And in the formula for B, since $\int yx\,dx\,.\,c = 0$, the inner integral may be estimated from any epoch. If the axis of y be also diametral, the calculation of the integrals may be further simplified by taking them for only one quadrant and doubling the results just stated.

We proceed to apply the foregoing theorem to the special question under consideration. The probability of five random points determining a convex pentagon on the particular surface proposed will be of precisely the same value if the diagram be supposed to be first projected orthogonally upon any plane.



Assume the relative position of the plane to be such that the rectangle is projected into the square ABCD; then the lines of section, EF, E′F′ in which the corners are cut off will still be parallel to the diagonal AC, and they will be bisected in m, m' by the other diagonal BD. The remaining surface in which the five random points are supposed to be taken is EAE′F′CF, and AC and DB are evidently the principal axes to this surface. We therefore take them as the axes of coordinates. As the probability in the question is dependent on relative magnitudes only, we may obviously assume AG = GB = 1; Em = mB = α; and x, y the coordinates of any point in the boundary of

surface. The form of the surface being symmetrical with respect to both axes, we shall avail ourselves of the simplification pointed out in the latter part of the note to the theorem, and consider only the quadrant AGmE, in which the limits are evidently $x = 0$ to $1-a$, and $y = 1-x$ throughout. Hence the following integrations :—

$$\int yx^2\,dx = \int_0^{1-a}(1-x)\,x^2\,dx = \frac{(1-a)^3}{3} - \frac{(1-a)^4}{4} = (1-a)^3\,\frac{1+3a}{12} \quad\ldots\ldots(1),$$

$$\int y^3\,dx = \int_0^{1-a}(1-x)^3\,dx = \frac{1-a^4}{4} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\int y^3x^2\,dx = \int_0^{1-a}(1-x)^3\,x^2\,dx = \frac{(1-a)^3}{3} - \frac{3\,(1-a)^4}{4} + \frac{3\,(1-a)^5}{5} - \frac{(1-a)^6}{6}$$

$$= (1-a)^3\,\frac{1+3a+6a^2+10a^3}{60} \quad\ldots\ldots\ldots\ldots(3),$$

$$yx\,dx\int y^2\,dx = \int_0^{1-a}(1-x)\,x\,dx\int(1-x)^2\,dx$$

$$= \int_0^{1-a}(1-x)\,x\,dx\left(x-x^2+\frac{x^3}{3}\right) = (1-a)^3\,\frac{4+12a+9a^2+5a^3}{90}\ldots\ldots(4).$$

Also $\quad \frac{1}{4}\mathrm{M} = (1-a)\,a + \frac{1}{2}(1-a)^2 = \frac{1}{2}(1-a)\,(1+a)\ldots\ldots\ldots\ldots\ldots(5).$

Therefore, by the theorem, we have

$$h^2 = \frac{(1)}{(5)} = \frac{(1-a)^2}{6}\cdot\frac{1+3a}{1+a},\quad k^2 = \frac{1}{3}\,\frac{(2)}{(5)} = \frac{1+a^2}{6};$$

$$\mathrm{A} = \frac{(3)}{(5)} = \frac{(1-a)^2}{30}\cdot\frac{1+3a+6a^2+10a^3}{1+a},$$

$$\mathrm{B} = \frac{(4)}{(5)} = \frac{(1-a)^2}{45}\cdot\frac{4+12a+9a^2+5a^3}{1+a},$$

$$\mathrm{C} = \frac{1}{3}\mathrm{A} + 3\mathrm{B} = \frac{(1-a)^2}{18}\cdot\frac{5+15a+12a^2+8a^3}{1+a},$$

$$h^2k^2 = \frac{(1-a)^2}{36}\cdot\frac{1+3a+a^2+3a^3}{1+a},$$

$$\mathrm{C} + h^2k^2 = \frac{(1-a)^2}{36}\cdot\frac{11+33a+25a^2+19a^3}{1+a};$$

$$(\Delta) = \frac{\mathrm{C}+h^2k^2}{\mathrm{M}^2} = \frac{11+33a+25a^2+19a^3}{144\,(1+a)^3} = \frac{11}{144} - \frac{a^2\,(1-a)}{18\,(1+a)^3},$$

$$(\Delta^2) = \frac{3}{2}\,\frac{h^2k^2}{\mathrm{M}^2} = \frac{1+3a+a^2+3a^3}{96\,(1+a)^3} = \frac{1}{96} - \frac{a^2\,(1-a)}{48\,(1+a)^3}.$$

$$(\Delta)-(\Delta^2) = \frac{19}{288} - \frac{5a^2\,(1-a)}{144\,(1+a)^3},\quad (\Delta)-2\,(\Delta^2) = \frac{1}{18} - \frac{a^2\,(1-a)}{72\,(1+a)^3};$$

therefore $\quad 1-10\,(\Delta)+10\,(\Delta^2) = \dfrac{49}{144} + \dfrac{25a^2\,(1-a)}{72\,(1+a)^3},$

$$10\,(\Delta)-20\,(\Delta^2) = \frac{5}{9} - \frac{5a^2\,(1-a)}{36\,(1+a)^3},\quad 10\,(\Delta^2) = \frac{5}{48} - \frac{5a^2\,(1-a)}{24\,(1+a)^3}.$$

Hence, if we put $\qquad \mu = \dfrac{a^2\,(1-a)}{72\,(1+a)^3},$

the values of the several probabilities come out as follows :—

Reentrant quadrilateral $\frac{11}{36}-16\mu$

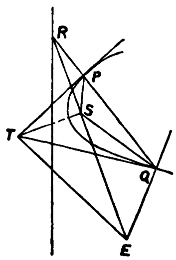
Convex quadrilateral	$\frac{25}{36}+16\mu$	One reentrant point	$\frac{5}{9}-10\mu$
Convex pentagon	$\frac{49}{144}+25\mu$	Two reentrant points	$\frac{5}{48}-15\mu$

The probability of a convex pentagon is a maximum when the function μ is a maximum; and, putting $\frac{d\mu}{d\alpha}=0$, we find $\alpha=\frac{1}{2}$, giving also $\mu=\frac{1}{1944}$. Thus it will appear that the lines of section suggested in the question will bisect the sides of the given rectangle; and, in such case, it is remarkable that the remaining surface will admit of being orthogonally projected from or into a regular hexagon. Indeed it may be concluded generally that, if five random points are taken on any polygonal surface, the probability that they shall be the apices of a convex pentagon will be a maximum when the surface is either a regular polygon or a projection of a regular polygon.

5913. (By C. Taylor, M.A.)—Prove that (1) a normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord; and (2) the polar of the middle point of the chord meets the focal vector to its point of concourse with the directrix upon the normal at its further extremity.

I. *Solution by* G. Turriff, M.A.; R. Knowles, L.C.P.; *and others.*

1. Let S be the focus, and join ST, SR. Then the foot R of the chord PQ, and the pole T, subtend a right angle at S (Taylor's *Geometry of Conics*, Art. 112). Hence a circle will go round RPST. Therefore $\angle$ TRS = TPS = STQ; hence TQ touches the circle, and is therefore at right angles to RT.



2. Let RS produced meet the normal at Q in E. Then, since a circle will go round STEQ, $\angle$ ETQ = ESQ = RSP (since SR bisects the supplement of PSQ) = RTP (in same segment of RPST) = TQP, by (1); hence ET is parallel to PQ.

II. *Solution by* C. F. D'Arcy, M.A.; D. Edwardes; *and others.*

1. Let (ξ, η) be the pole of the chord of $y^2-4ax=0$, normal at (x', y'); then $x'=-(2a+\xi)$, and $ay'=(2a+\xi)\,\eta$, which gives $(2a+\xi)\,\eta^2+4a^3=0$ as the locus of the poles of normal chords (Quest. 5088); and if (α, β) is its point of concourse with the directrix

$$\xi-\alpha=a+\xi,\quad \eta-\beta=(a+\xi)\,\xi\eta \div 2a^2,$$

whence
$$\eta-\beta : \xi-\alpha = \xi\eta : 2a^2.$$

But the equation to the second normal is

$$y+\frac{\xi\eta}{2a^2}\left(-x+2a+\frac{\xi^2}{2a+\xi}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

2. The polar of the middle point of the normal-chord is

$$2a^2(y-\eta)+(2a+\xi)\,\eta\,(x-\xi)=0 \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

and the focal vector to $\alpha\beta$ is

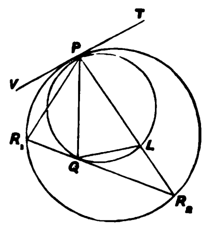
$$4a^3y+(a-\xi)(za+\xi)(x-a)=0 \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}$$

Now (i.) $\times\,(2a+\xi)$ + (ii.) $\times\,(2a-\xi)$ − (iii.) $\times$ 2 vanishes identically.

5380. (By W. Gallatly, B.A.)—If a circle A touches internally another circle B at P, and a tangent to A at the point Q intersect B in R_1, R_2, prove that $\angle R_1PQ = \angle R_2PQ$.

[This reciprocates into the following:—S is the common focus of two conics having contact of the second order at P_1; if from a point Q on the outer conic, two tangents QR_1, QR_2 be drawn to the inner conic intersecting the common tangent at P in the points R_1, R_2, and if the tangent at Q to the outer conic intersect the tangent at P in T; then TR_1, TR_2 subtend equal angles at the focus.]

Solution by W. J. Macdonald, M.A.; Donald Macalister, B.A., B.Sc.; *and others.*



At P draw the common tangent VPT, and join QL; then $\angle PR_1Q = TPL = PQL$,

and $\angle PQR_1 = PLQ$,

therefore $\angle R_1PQ = QPL$.

[This is a better proof of the theorem than the one given on p. 26 of Vol. XXIX. of the *Reprint.*]

5786. (By A. Martin, M.A.)—A square hole is cut through the centre of a sphere of radius r; show that the average volume removed is

$$\tfrac{1}{15}(23\sqrt{2}-28)\,\pi r^3.$$

I. *Solution by* E. B. Seitz.

Let x = the side of the hole, y = the distance of a plane parallel to the side of the hole from the centre of the sphere, u = the area of the section of the hole made by this plane, and V the required average.

Then $u = 2(r^2-y^2)\sin^{-1}\left\{\frac{x}{2(r^2-y^2)^{\frac{1}{2}}}\right\} + x(r^2-y^2-\frac{1}{4}x^2)^{\frac{1}{2}}$,

$$V = \frac{2\int_0^{r\sqrt{2}}\int_0^{\frac{1}{2}x} u\,dx\,dy}{\int_0^{r\sqrt{2}} dx} = \frac{\sqrt{2}}{r}\int_0^{r\sqrt{2}}\int_0^{\frac{1}{2}x} u\,dx\,dy$$

$$= \frac{\sqrt{2}}{6r}\int_0^{r\sqrt{2}}\Big[(12r^2x-x^3)\sin^{-1}\left\{\frac{x}{(4r^2-x^2)^{\frac{1}{2}}}\right\}$$
$$+16r^3\tan^{-1}\left(1-\frac{x^2}{2r^2}\right)^{\frac{1}{2}} - 4\pi r^3 + x^2(4r^2-2x^2)^{\frac{1}{2}}\Big]dx$$
$$= \tfrac{1}{12}\pi r^3(23\sqrt{2}-28).$$

II. *Solution by the* Proposer.

Let r = radius of the sphere, $m = \frac{1}{2}r\sqrt{2}$, $2w$ = a side of the square hole, V = volume removed, and Δ = *average* volume removed. Then, by the solution of Question 4257, we have

$$V = \tfrac{4}{3}w^2(r^2-2w^2)^{\frac{1}{2}} + \tfrac{4}{3}w(3r^2-w^2)\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right)^{\frac{1}{2}} - \tfrac{4}{3}r^3\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right),$$

$$\Delta = \int_0^m V dw \div \int_0^m dw$$

$$= \frac{\sqrt{2}}{r}\Big[\tfrac{1}{3}\sqrt{2}(6w^3-\tfrac{1}{2}r^2w)(\tfrac{1}{2}r^2-w^2)^{\frac{1}{2}} - \sqrt{2}\,r^4\sin^{-1}\left(\frac{w\sqrt{2}}{r}\right)$$
$$-\tfrac{8}{3}r^3w\sin^{-1}\left(\frac{w^2}{r^2-w}\right) + \tfrac{2}{3}(6r^2w^2-w^4+3r^4)\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right)^{\frac{1}{2}}\Big]_0^{\frac{1}{2}r\sqrt{2}},$$
$$= \tfrac{1}{12}\pi r^3(23\sqrt{2}-28).$$

5829. (By Professor Cochez.) — Trouver les conditions pour que l'équation du quatrième degré $Ax^4+Cx^2+Dx+F=0$ ait trois racines égales. On trouve les 2 conditions

$$\left(\frac{3}{2}\right)^3\left(\frac{D}{A}\right)^2 = 12\frac{CF}{A^2} = -\frac{C^3}{A^3}.$$

I. *Solution by* J. A. Kealy, M.A.; J. Young, B.A.; *and others.*

Let a be the root repeated three times, than other root is $-3a$; hence

$(x-a)^3(x+3a)$, $x^4+\frac{C}{A}x^2+\frac{D}{A}x+\frac{F}{A}$ are identical expressions.

Equating coefficients, we have $-6a^2 = \frac{C}{A}$, $8a^3 = \frac{D}{A}$, $-3a^4 = \frac{F}{A}$;

therefore $$216a^6 = -\frac{C^3}{A^3} = 12\frac{CF}{A^2} = \frac{27}{8}\left(\frac{D}{A}\right)^2.$$

II. *Solution by the* Rev. J. L. Kitchin, M.A; R. Graham, M.A.; *and others.*

Consider the equation $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$.

The conditions for its having three equal roots are

$$ae-4bd+3c^2=0,\quad ace+2bcd+ad^2-b^2e-c^3=0;$$

which can hardly be unknown, though I am unable to refer to any proof of them. An inspection of Prof. CAYLEY's expressions for these invariants in terms of the roots at once verifies them; or they may be obtained by transforming $ax^4+\ldots$ into $(a'x'+4b')x^3$, or thus:

Assume $ax^4+\ldots=a(x+\alpha)(x+\beta)^3$;

then $4b=a(3\beta+\alpha),\quad 6c=a(3\beta^2+3\alpha\beta),$

$4d=a(\beta^3+3\alpha\beta^2),\quad e=a\alpha\beta^3;$

whence $4ae=4a^2\alpha\beta^3,\quad -16bd=-3a^2\alpha^2\beta^2-10a^2\alpha\beta^3-3a^2\beta^4,$

and $12c^2=3a^2\alpha^2\beta^2+6a^2\alpha\beta^3+3b^4,$

giving $4(ae-4bd+3c^2)$ identically equal 0.

Again $64\begin{vmatrix} a & b & c\\ b & c & d\\ c & d & e\end{vmatrix}=\begin{vmatrix} 4 & \alpha+3\beta & 2(\alpha+\beta)\beta\\ \alpha+3\beta & 2(\alpha+\beta)\beta & (3\alpha+\beta)\beta^2\\ 2(\alpha+\beta)\beta & (3\alpha+\beta)\beta^2 & 4\alpha\beta^3\end{vmatrix}$ in which, if the first column be multiplied, and the third divided, by β, the sum of the first and third columns is equal to twice the second; and the determinant consequently vanishes identically.

Putting $a=12A,\ b=0,\ c=2C,\ d=3D,\ e=12F$, the conditions, as found above, are $12AF+C^2=0,\quad 72ACF-27AD^2-2C^3=0.$
Subtracting the first, multiplied by 6C, from the second,

$$27AD^2+8C^3=0;$$

and this last, with the first, are the conditions as given in the Question.

5867. (By the Rev. W. A. WHITWORTH, M.A.) — Construct 40 different right-angled triangles having the common hypothenuse 32045, and all their sides integers.

I. *Solution by the* PROPOSER.

On the common hypothenuse = 32045 describe (Euclid I. 22) triangles having their other two sides equal to (i.) the first and last, (ii.) the 2nd and 79th, (iii.) the 3rd and 78th, and so on, of the 80 numbers registered in the subjoined table. The forty triangles thus described will be found to satisfy the test of Euclid I. 48. They are therefore the triangles required.

716	7259	12325	18291	23067	27004	29848	31323
1363	7656	12920	19227	23205	27132	29920	31603
2277	7888	13572	19552	23693	27347	30381	31668
2400	8283	15080	19795	24124	27608	30420	31800
3045	8580	15708	20300	24205	27813	30821	31824
3757	8772	15916	21000	24795	27931	30875	31900
3955	10075	16269	21093	25200	28275	30956	31955
4901	10192	16704	21576	25389	29029	31059	31964
5304	11475	17051	22100	25636	29325	31117	32016
6764	11661	17253	22244	26312	29580	31212	32037

II. *Solution by the* Rev. G. H. HOPKINS, M.A.

Fermat has proved that every prime number of the form $4N+1$ is the sum of two square numbers; consequently every such prime number is the hypothenuse of a Pythagorean triangle.

Mr. Whitworth, in a paper on Pythagorean triangles (read before the Literary and Philosophical Society of Liverpool, Feb., 1875), has proved—The product of n prime hypothenuses, all different, will be itself the hypothenuse of 2^{n-1} Pythagorean triangles.

The prime factors of 32045 are 5, 13, 17, 29, all being of the form $4N+1$; hence the hypothenuses arising directly from 32045 will be 2^{4-1}.

Next, take each of the primes singly as factors of all the sides of the triangles, found by the product of the remaining three factors, viz., 5×6409, 13×2465, 17×1885, 29×1105; and we obtain $4\times 2^{3-1}$ hypothenuses.

Thirdly, take the product of any two of the primes as factors of all the sides of the triangles found by the product of the remaining two factors, viz., 65×481, 85×377, 221×145, 481×65, 377×85, 145×221. By this arrangement, we obtain $\frac{4\,.\,3}{1\,.\,2}\times 2^{2-1}$ hypothenuses.

Lastly, take the product of any three of the primes as factors of all the sides of the triangles formed by the remaining factors, viz., 6409×5, 2465×13, 1885×17, 1105×29, this gives $\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\times 2^{1-1}$ hypothenuses.

In all, we obtain

$$2^{4-1}+4\,.\,2^{3-1}+\frac{4\,.\,3}{1\,.\,2}\,.\,2^{2-1}+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,.\,2^{1-1},$$

or
$$\tfrac{1}{2}\left(2^4+4\,.\,2^3+\frac{4\,.\,3}{1\,.\,2}\,.\,2^2+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,.\,2+1-1\right),$$

or
$$\tfrac{1}{2}(3^4-1),\ \text{or}\ 40.$$

The general formula for a number with n prime factors of the form $4N+1$ is easily inferred from the above example; the number of hypothenuses which such a number would give is $\frac{1}{2}(3^n-1)$.

To obtain the sides which correspond with the hypothenuse 32045, I must refer to the paper above mentioned, Art. 16, or to HEWLETT's Fifth Edition of EULER's *Algebra*, p. 390; and, to work out a single example, let

$$x^2+y^2 = 32045 = 5\,.\,13\,.\,17\,.\,29 = (2^2+1^2)(3^2+2^2)(4^2+1^2)(5^2+2^2),$$

$$(x+yi)(x-yi) = (2+i)(2-i)(3+2i)(3-2i)(4+i)(4-i)(5+2i)(5-2i).$$

Equating $x+iy$, where as usual, $i=\sqrt{-1}$, to four of the factors on the right-hand side of the identity, we can do it in 2^3 different ways. Take any one of these, thus

$$x+yi = (2+i)(3-2i)(4-i)(5+2i) = 179+2i,$$

or
$$x = 179 \text{ and } y = 2.$$

The sides given by these will be 179^2-2^2 or 32037 and $2\times 179\times 2$ or 716. Similarly for other values.

The forty sets of sides which coincide with 32045 are

2277 : 31964.	17253 : 27004.	23067 : 22244.	31323 : 6764.
8283 : 30956.	21093 : 24124.	27813 : 15916.	32037 : 716.
1363 : 32016.	23693 : 21576.	27347 : 16704.	31117 : 7656.
7259 : 31212.	17051 : 27132.	27931 : 15708.	30821 : 8772.
11661 : 29848.	18291 : 26312.	25389 : 19552.	30381 : 10192.
3955 : 31800.	19795 : 25200.	24205 : 21000.	31955 : 2400.
10075 : 30420.	30875 : 8580.	16269 : 27608.	31059 : 7888.
11475 : 29920.	29325 : 12920.	4901 : 31668.	29029 : 13572.
3045 : 31900.	24795 : 20300.	3757 : 31824.	31603 : 5304.
19227 : 25636.	12325 : 29580.	28275 : 15080.	23205 : 22100.

These four groups are obtained from the four different arrangements of the factors of the given number as given above.

If it be required to find the number of Pythagorean triangles, not necessarily in their lowest terms, which have 32045 for one of the sides, we can proceed as follows :—

$x^2-y^2 = 5 \, . \, 13 \, . \, 17 \, . \, 29$, this equation gives 2^{4-1} solutions.

$x^2-y^2 =$ any three of the factors, this would give in all $4 \, . \, 2^{3-1}$.

$x^2-y^2 =$ any two of the factors, this would give in all $\dfrac{4 \, . \, 3}{1 \, . \, 2} \, . \, 2^{2-1}$.

$x^2-y^2 =$ any one of the factors, this would give in all $\dfrac{4 \, . \, 3 \, . \, 2}{1 \, . \, 2 \, . \, 3} \, . \, 2^{1-1}$.

The sum of these will be $\frac{1}{2}(3^4-1)$ or 40.

Also, if it be required to find the number of Pythagorean triangles having 2×32045 for one of the sides, which is always of the form $2ab$, we shall have again the number $\frac{1}{2}(3^4-1)$ or 40; a particular case of the general number $2a_1 a_2 a_3 \ldots a_n$, which I proved in the *Educational Times*, about five or six years back, to give $\frac{1}{2}(3^n-1)$ sides of the form $2ab$.

5911 (By the Rev. W. A. Whitworth, M.A.)—Show that the 80 acute angles of the 40 triangles of Question 5867 are obtained by rejecting all complete right angles from the values of the expression

$$180^\circ + i\alpha + i'\beta + i''\gamma + i'''\delta\,;$$

where i, i', i'', i''' may each have any of the values $-1, 0, +1$; and where

$$\alpha = 36^\circ\, 52'\ldots \quad \beta = 22^\circ\, 37'\ldots \quad \gamma = 28^\circ\, 4'\ldots \quad \delta = 43\; 36'\ldots$$
$$= \tan^{-1}\tfrac{3}{4}, \quad = \tan^{-1}\tfrac{5}{12}, \quad = \tan^{-1}\tfrac{8}{15}, \quad = \tan^{-1}\tfrac{20}{21}.$$

[It may be remarked that, since each of the four i, i', i'', i''' is susceptible of three values, the expression will give 3^4 or 81 angles. One of these is, however, zero; the remaining 80 are the angles required.]

Solution by the Rev. G. H. Hopkins, M.A.

Let Z be such a number as 32045, which has all of its prime factors $z_1, z_2, z_3 \ldots z_n$ of the form $4N+1$.

Then, if $\theta_1\, \theta_2\, \theta_3 \ldots \theta_n$ be each of them one of the angles of the n Pythagorean triangles which have $z_1, z_2, z_3 \ldots z_n$ for the hypothenuses, the sides

will be $z_1\cos\theta_1,\ z_2\cos\theta_2,\ z_3\cos\theta_3 \dots z_n\cos\theta_n,$

$z_1\sin\theta_1,\ z_2\sin\theta_2,\ z_3\sin\theta_3 \dots z_n\sin\theta_n.$

Taking all the expressions similar to $z_r(\cos\theta_r \pm i\sin\theta_r)$, or, as it may be written, $z_r(\cos\epsilon_r\theta_r + i\sin\epsilon_r\theta_r)$, when i is $\sqrt{-1}$ and $\epsilon_r = \pm 1$; and forming the product of all these expressions, we obtain

$$Z\left\{\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots + \epsilon_n\theta_n) + i\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots \epsilon_n\theta_n\right\},$$

which shows that, if Z be a given hypothenuse, then the sides are

$$Z\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots + \epsilon_n\theta_n) \text{ and } Z\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots + \epsilon_n\theta_n).$$

Thus far we have taken $\epsilon_1, \epsilon_2, \epsilon_3 \dots \epsilon_n$ each equal to ± 1; their values may be extended by putting them also equal to 0; for if $\epsilon_r = 0$, then z_r would be a common factor for all the sides of one triangle; hence $\epsilon_1, \epsilon_2, \epsilon_3 \dots \epsilon_n$ can have any of the values $-1, 0, +1$; we must except the case where they are all equal to 0 for obvious geometrical reasons. We have, then, n quantities which can each of them have any one of 3 values; the number of combinations of these will be 3^n; and, allowing for the exception mentioned, we have $3^n - 1$, which gives $\frac{1}{2}(3^n - 1)$ triangles.

By hypothesis, all the n sides, of which $z_r\cos\theta_r$, $z_r\sin\theta_r$ are types, are integers; the two terms in the expression $z_r(\cos_r\theta_r + i\sin\epsilon_r\theta_r)$ are integers, it follows that in the product of these n expressions the two terms will be integers; therefore the sides of the $\frac{1}{2}(3^n - 1)$ triangles which have z for their hypotheneuse will be the $\frac{1}{2}(3^n - 1)$ integral values of

$$Z\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots \epsilon_n\theta_n) \text{ and } Z\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots \epsilon_n\theta_n).$$

If the angle in the bracket is written in the more general form

$$k\,180^\circ + \epsilon_1\theta_1 + \epsilon_2\theta_2 + \dots \epsilon_n\theta_n,$$

giving k such a value that the angle is not negative, and throwing out all right angles, we shall have the acute angles of the triangles. And if

$x_r^2 + y_r^2 = z_r^2$, then $\qquad \theta_r = \tan^{-1}\dfrac{x_r}{y_r}.$

Applying this to the particular number 32045, the prime factors are 5, 13, 17, 29; the sides corresponding to these four hypothenuses are 3, 4; 5, 12; 8, 15; 20, 21; and if $\alpha, \beta, \gamma, \delta$ be one of the acute angles in these four right-angled triangles

$$\alpha = \tan^{-1}\tfrac{3}{4},\ \beta = \tan^{-1}\tfrac{5}{12},\ \gamma = \tan^{-1}\tfrac{8}{15},\ \delta = \tan^{-1}\tfrac{20}{21}.$$

Mr. Whitworth gives the values of these angles, and as their sum lies between 90° and 180°, in the general expression we may put $k = 1$, or all the acute angles of the $\frac{1}{2}(3^4 - 1)$ triangles are contained in the formula

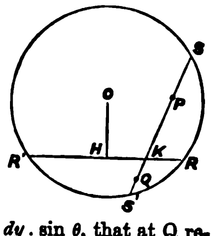
$$180^\circ + \epsilon_1\alpha + \epsilon_2\beta + \epsilon_3\gamma + \epsilon_4\delta.$$

For various obscure points in this Solution, I must refer to my solution of Quest. 5867, and Mr. Whitworth's pamphlet there mentioned.

5872 & 5891. (By E. B. Seitz and S. Roberts, M.A.)—A line is drawn at random across a circle, and a point is taken at random in each segment; show that (1) the average distance between the points is $\frac{45}{128}\pi r$, and (2) the average length of the chord is $\frac{75}{128}\pi r$.

I. *Solution by* Samuel Roberts, M.A.

1. Let RR′SS′ be a circle with centre O and radius r, R′R a random line dividing the circle into the segments R′RS, R′RS′, represented by A, B respectively. Take P a random point in A, and Q a random point in B, and let the chord through them meet the circle in S, S′, and the random line in K. Draw OH $= k$ to the middle point H of RR′; and put HR $= l$, HK $= x$, KP $= y$, KS $=$ Y, KQ $= z$, KS′ $=$ Z, $\angle$RKP $= \theta$. We need only consider random lines parallel to R′R, for these become any other parallel system by turning the whole figure through an angle.



The element of the circular surface at P is $dx \,.\, dy \,.\, \sin\theta$, that at Q referred to the point P is $(y+z)\; dz\; d\theta$, and the required average is therefore

$$\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi}\int_0^{Y}\int_0^{Z} (y+z)^2\, dk \,.\, dx \,.\, \sin\theta\, d\theta \,.\, dy \,.\, dz \div \int_{-r}^{+r} \text{A} \,.\, \text{B} \,.\, dk \quad \ldots\ldots(\alpha),$$

The integral in the denominator is most simply obtained by the chance that P, Q are on opposite sides of the random line (Williamson's *Int. Calc.*, p. 340, ex. 40), viz.,

$$2\int_{-r}^{+r} \text{A} \,.\, \text{B} \,.\, dk \div 2r \,.\, (\pi r^2)^2 = \frac{128}{45\pi^2}; \quad \therefore \int_{-r}^{+r} \text{A} \,.\, \text{B} \,.\, dk = \frac{128}{45} r^5.$$

For (α) we have,

$$\frac{15}{256 r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi} \text{YZ}\left\{2\,(\text{Y}+\text{Z})^2 - \text{YZ}\right\} dk \,.\, dx \,.\, \sin\theta \,.\, d\theta.$$

But $\qquad \text{Y} = \left\{r^2 - (x\sin\theta + k\cos\theta)^2\right\}^{\frac{1}{2}} + k\sin\theta - x\cos\theta,$

and $\qquad \text{Z} = \left\{r^2 - (x\sin\theta + k\cos\theta)^2\right\}^{\frac{1}{2}} - k\sin\theta + x\cos\theta;$

$$\therefore \frac{15}{256 r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi} \left\{7\,(l^2-x^2)^2 + 8\,(x\cos\theta - k\sin\theta)^2\,(l^2-x^2)\right\} dk\, dx \sin\theta\, d\theta$$

$$= \frac{15}{128 r^5}\int_{-r}^{+r}\int_{-l}^{+l} \left\{7\,(l^2-x^2)^2 + \tfrac{8}{3}\,(l^2-x^2)\,x^2 + \tfrac{16}{3}\,(l^2-x^2)\,k^2\right\} dk\, dx$$

$$= \frac{15}{64 r^5}\int_{-r}^{+r} \left(\frac{184 l^5 + 160 l^3 k^2}{45}\right) dk = \frac{45}{128}\pi r.$$

2. Here the required average is

$$\frac{\displaystyle\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi}\int_0^{Y}\int_0^{Z} (\text{Y}+\text{Z})\,(y+z)\, dk\, dx \sin\theta\, d\theta\, dy\, dz}{\displaystyle\int_{-r}^{r} \left(\pi r^2 + kl - r^2\cos^{-1}\frac{k}{r}\right)\left(r^2\cos^{-1}\frac{k}{r} - kl\right) dk}$$

$$= \tfrac{1}{2}\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi} \text{YZ}\,(\text{Y}+\text{Z})^2\, dk\, dx \sin\theta\, d\theta \div \left(\tfrac{128}{45} r^5\right).$$

But $\quad Y = \{r^2-(x\sin\theta+k\cos\theta)^2\}^{\frac{1}{2}}+k\sin\theta-x\cos\theta,$

and $\quad Z = \{r^2-(x\sin\theta+k\cos\theta)^2\}^{\frac{1}{2}}-k\sin\theta+x\cos\theta\,;$

hence the required average is

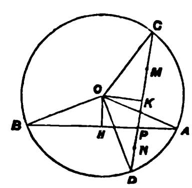
$$\frac{45}{64r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi}(l^2-x^2)\{r^2-(x\sin\theta+k\cos\theta)^2\}\,dk\,dx\,\sin\theta\,d\theta$$

$$=\frac{45}{32r^5}\int_{-r}^{+r}\int_{-l}^{+l}\{(l^2-x^2)\,r^2-\tfrac{2}{3}(l^2-x^2)\,x^2-\tfrac{1}{3}(l^2-x^2)\,k^2\}\,dk\,dx$$

$$=\frac{1}{16r^5}\int_{-r}^{+r}\left\{\frac{13l^5+10l^3k^2}{8}\right\}dk=\tfrac{75}{128}\pi r.$$

II. *Solution by* E. B. Seitz.

1. Let AB be the chord formed by the random line, M, N the random points, CD the chord through them, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD. Let OA $=r$, PM $=x$, MN $=y$, PC $=u$, PD $=v$, $\angle$ AOH $=\theta$, $\angle$ HOK $=\phi$, and $\angle$ COK $=\psi$. Then we have



$$CD = 2r\sin\psi,$$

$$\text{area seg. ADB} = r^2(\theta-\sin\theta\cos\theta),$$

$$\text{area seg. ACB} = r^2(\pi-\theta+\sin\theta\cos\theta),$$

$$u = r(\sin\psi+\cos\theta\,\mathrm{cosec}\,\phi-\cot\phi\cos\psi),$$

and $$v = r(\sin\psi-\cos\theta\,\mathrm{cosec}\,\phi+\cot\phi\cos\psi).$$

An element of the circle at M is $r\sin\psi\,d\psi\,dx$, at N it is $y\,d\phi\,dy$, and at H an element of the distance across the circle in the direction HO is $r\sin\theta\,d\theta$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$, and doubled; of ψ, $\theta-\phi$ and $\theta+\phi$ when $\phi<\theta$, and $\phi-\theta$ and $\phi+\theta$ when $\phi>\theta$. But in the integration with respect to ψ we will have powers of $\cos\psi$ only; hence, since

$$\cos(\theta-\phi) = \cos(\phi-\theta),$$

we may take $\theta-\phi$ and $\theta+\phi$ as the limits of ψ for all values of ϕ. The limits of x are 0 and u; and of y, x and $v+x$.

Therefore the average distance between the points is

$$\Delta=\frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x} r\sin\theta\,d\theta\,d\phi\;r\sin\psi\,d\psi\,dx\,y^2\,dy}{\int_0^{\frac{1}{2}\pi} r^5(\theta-\sin\theta\cos\theta)(\pi-\theta+\sin\theta\cos\theta)\sin\theta\,d\theta}$$

$$=\frac{45}{32r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x}\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi\,dx\,y^2\,dy$$

$$=\frac{15}{32r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}[(v+x)^3-x^3]\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi\,dx$$

$$= \frac{15}{128r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\left\{(u+v)^4-(u^4+v^4)\right\}\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi$$

$$= \frac{r}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(10-3\sin^2\theta-10\cos^2\phi+12\sin^2\theta\cos^2\phi)\sin^4\theta\sin\phi\,d\theta\,d\phi$$

$$= \frac{r}{12}\int_0^{\frac{1}{2}\pi}(20+3\sin^2\theta)\sin^4\theta\,d\theta = \tfrac{45}{128}\pi r.$$

2. The average length of the chord through the points is

$$L = \frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x} 2r\sin\psi\; r\sin\theta\,d\theta\,d\phi\; r\sin\psi\,d\psi\,dx\,y\,dy}{\int_0^{\pi} r^5(\theta-\sin\theta\cos\theta)(\pi-\theta+\sin\theta\cos\theta)\sin\theta\,d\theta}$$

$$= \frac{45}{16r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x}\sin\theta\sin^2\psi\,d\theta\,d\phi\,d\psi\,dx\,y\,dy$$

$$= \frac{45}{32r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\left\{(v+x)^2-x^2\right\}\sin\theta\sin^2\psi\,d\theta\,d\phi\,d\psi\,dx$$

$$= \frac{45}{16r}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}uv\sin\theta\sin^3\psi\,d\theta\,d\phi\,d\psi$$

$$= \frac{3r}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(5-\sin^2\theta-5\cos^2\phi+6\sin^2\theta\cos^2\phi)\sin^4\theta\sin\phi\,d\theta\,d\phi$$

$$= \frac{r}{4}\int_0^{\frac{1}{2}\pi}(10+3\sin^2\theta)\sin^4\theta\,d\theta = \tfrac{75}{128}\pi r.$$

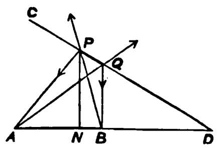
5904. (By Professor MINCHIN, M.A.)—Iron filings are scattered over a sheet of paper on which rests a magnet. Prove that the locus of those filings which dip towards the same point in the line of the magnet is a circle. [Magnetism is supposed to be concentrated at the poles.]

I. *Solution by* H. L. ORCHARD, B.A., L.C.P.; Rev. J. L. KITCHIN, M.A.; E. RUTTER; *and others.*

If P, P′ are the poles of, and Q the fixed point on, the magnet, while O is one of the filings dipping towards Q, the condition $\frac{\sin POQ}{OP^2} = \frac{\sin P'OQ}{OP'^2}$

must be fulfilled; but, obviously, $\sin POQ : \sin P'OQ = \dfrac{PQ}{OP} : \dfrac{P'Q}{OP'}$ so that $OP : OP' = (PQ)^{\frac{1}{3}} : (P'Q)^{\frac{1}{3}}$, and the locus of O is the circle whose diameter is the segment between points in which PP′ is divided in the subtriplicate ratio of PQ to P′Q.

II. *Solution by* Rev. G. H. Hopkins, M.A.; Prof. Evans, M.A.; *and others.*



This theorem resolves itself into the following:—A and B are two centres of force which act upon P and Q, the extremities of a small rigid bar, in such a way that A attracts B and repels Q, and B attracts Q and repels P; the law of force being some power of the distance (d^n); then, if PQ is at rest, with its direction cutting AB produced in a fixed point D, when PQ becomes very small, the locus of its positions of equilibrium will be a circle.

The small rigid bar PQ is in equilibrium under the action of the forces from A and B; the sum of the moments of the forces acting on P round Q is

$$\phi(AB)\sin APC - \phi(BP)\sin BPD = 0,$$

where $\phi(d)$ is put for the law of force. Therefore

$$\frac{\phi(AP)}{\phi(BP)} = \frac{\sin(PBA - PDA)}{\sin(PAB + PDA)} = \frac{\sin PBA - \cos PBA \tan PDA}{\sin PAB + \cos PAB \tan PDA}$$

$$= \frac{\dfrac{PN}{PB} - \dfrac{BN}{PB}\cdot\dfrac{PN}{DN}}{\dfrac{PN}{PA} + \dfrac{PN}{PA}\cdot\dfrac{PN}{DN}};$$

therefore $$\frac{PB\phi(AP)}{PA\phi(BP)} = \frac{1 - \dfrac{BN}{DN}}{1 + \dfrac{AN}{DN}} = \frac{DB}{DA} = \text{a constant ratio.}$$

Assume that the law of force is as the n^{th} power of the distance, then

$$\frac{PB\phi(AP)}{PA\phi(BP)} \propto \left(\frac{AP}{BP}\right)^{n-1}, \quad \text{therefore } \frac{AP}{BP} = \text{a constant } (k).$$

If the middle point of AB be the origin, and AB lie in the axis of x,

$$\frac{y^2 + (x+a)^2}{y^2 + (x-a)^2} = k^2, \quad \text{or} \quad y^2 + x^2 + 2\,\frac{1+k^2}{1-k^2}\,ax + a^2 = 0 \text{ is the required locus.}$$

This locus is independent of the value of n, and is therefore true for magnetism where $n = -2$.

5627. (By Professor Townsend, F.R.S.)—A cylinder of revolution of uniform density being supposed to float, with axis vertical, in a gravitating fluid of greater density; show, by any method, that the coaxal spheroid

of revolution, the square of whose equatoreal is double of that of its polar radius, will intersect its surface in the limiting circle of floatation consistent with the stability of its equilibrium under the action of gravity.

Solution by the PROPOSER.

Denoting by h and r the height and radius of the cylinder, and by z the depth of its immersion in the fluid; then, since, on well-known principles, the elevation of the metacentre corresponding to the position of floatation above the centre of inertia of the cylinder $= \frac{r^2}{4z} - \frac{h-z}{2}$, therefore, for the limiting value of z consistent with the stability of its equilibrium under the action of gravity, $2z(h-z) = r^2$; and therefore, &c.

5772. (By C. LEUDESDORF, M.A.)—If the areas of the pedals of any given closed curve with regard to four points A, B, C, P in its plane, taken successively as pedal origin, be denoted by (A), (B), (C), (P) respectively, prove that $\quad (P) = x(A) + y(B) + z(C) + \frac{1}{2}\pi t^2,$
where x, y, z are the areal coordinates of P referred to the triangle ABC, and t is the length of the tangent from P to the circle around ABC.

Solution by W. J. C. SHARP, B.A.; J. A. KEALY, M.A.; *and others.*

If S be the area of the minimum pedal, the origin of which is chosen as origin of rectangular Cartesian coordinates, and A, B, C, P denote the points (ab), (a_1b_1), (a_2b_2), P (xy), we have (Williamson's *Int. Calc.*, p. 183)

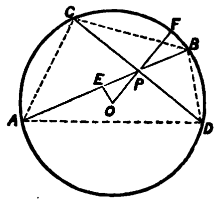
$$(A)-S = \tfrac{1}{2}\pi(a^2+b^2),\quad (B)-S = \tfrac{1}{2}\pi(a_1^2+b_1^2),$$
$$(C)-S = \tfrac{1}{2}\pi(a_2^2+b_2^2),\quad (P)-S = \tfrac{1}{2}\pi(\xi^2+\eta^2);$$

$$\begin{vmatrix} \xi^2+\eta^2 & \xi & \eta & 1 \\ a^2+b^2 & a & b & 1 \\ a_1^2+b_1^2 & a_1 & b_1 & 1 \\ a_2^2+b_2^2 & a_2 & b_2 & 1 \end{vmatrix} = \Delta t^2, \quad \text{therefore} \quad \begin{vmatrix} (P) & \xi & \eta & 1 \\ (A) & a & b & 1 \\ (B) & a_1 & b_1 & 1 \\ (C) & a_2 & b_2 & 1 \end{vmatrix} = \tfrac{1}{2}\pi\Delta t^2;$$

therefore $(P)\Delta = (A)\Delta(BPC) + (B)\Delta(APC) + (C)\Delta(APB) + \frac{1}{2}\pi\Delta\mu^3$,
where Δ stands for the triangle ABC, therefore, &c.

5168. (By ARTEMAS MARTIN, M.A.)—Through a point taken at random in the surface of a circle, two chords are drawn, one at random, and the other at right angles to the radius through that point; show that the average area of the quadrilateral formed by joining the ends of the chords is one-fourth of the area of the circle.

Solution by the PROPOSER; Professor NASH, M.A.; *and others.*



Let O be the centre of the circle, P the random point in its surface, AB the random chord, and CD the chord at right angles to the radius through P. Draw OE perpendicular to AB. Let OP $= x$ and $\angle$ APO $= \phi$; then

$$\angle \text{APC} = \tfrac{1}{2}\pi - \phi, \quad \text{CD} = 2(r^2 - x^2)^{\frac{1}{2}},$$
$$\text{AB} = 2(r^2 - x^2 \sin^2\phi)^{\frac{1}{2}};$$

and the area of the quadrilateral

$$\text{ACBD} = \text{AB} \times \text{CD} \times \tfrac{1}{2}\sin \text{APC} = 2(r^2-x^2)^{\frac{1}{2}}(r^2 - x^2\sin^2\phi)^{\frac{1}{2}}\cos\phi.$$

If Δ be the average area required, we have

$$\Delta = \frac{2\int_0^r\int_0^{\frac{1}{2}\pi}(r^2-x^2)^{\frac{1}{2}}(r^2-x^2\sin^2\phi)^{\frac{1}{2}}\cos\phi \times 2\pi x\,dx\,d\phi}{\int_0^r\int_0^{\frac{1}{2}\pi} 2\pi x\,dx\,d\phi}$$

$$= \frac{8}{\pi r^2}\int_0^r\int_0^{\frac{1}{2}\pi}(r^2-x^2)^{\frac{1}{2}}(r^2-x^2\sin^2\phi)^{\frac{1}{2}}\,x\,dx\cos\phi\,d\phi$$

$$= \frac{4}{\pi r^2}\int_0^r (r^2-x^2)\,x\,dx + \frac{4}{\pi}\int_0^r (r^2-x^2)^{\frac{1}{2}}\sin^{-1}\left(\frac{x}{r}\right)dx$$

$$= \frac{r^2}{\pi} + \frac{1}{\pi}\left[2x(r^2-x^2)^{\frac{1}{2}}\sin^{-1}\left(\frac{x}{r}\right) + r^2\left(\sin^{-1}\frac{x}{r}\right)^2 - x^2\right]_0^r$$

$$= \tfrac{1}{4}\pi r^2.$$

5728. (By F. P. MATZ, M.A.)—An equilateral spherical triangle is (1) inscribed in, (2) circumscribed about, a circle of radius ϕ; show that the cosine of an angle of the triangle is $\dfrac{3\cos^2\phi - 1}{3\cos^2\phi + 1}$ in (1), and $\frac{3}{2}\cos^2\phi - 1$ in (2).

Solution by Rev. J. L. KITCHIN, M.A.; R. KNOWLES, L.C.P.; *and others.*

1. Let ABC be the spherical triangle; P the pole of the circle; and PD a perpendicular to AB. Denote the arc AP by ϕ, and let $\angle$ PAD $=$ PAC $= \frac{1}{2}\beta$. But the $\angle$ APD $= \frac{1}{3}\pi$; consequently

$$\cot\tfrac{1}{2}\beta = \cos\phi\tan\tfrac{1}{3}\pi = \cos\phi\sqrt{3}, \ \therefore\ \cos\beta = \frac{\cot^2\frac{1}{2}\beta - 1}{\cot^2\frac{1}{2}\beta + 1} = \frac{3\cos^2\phi - 1}{3\cos^2\phi + 1}.$$

2. Let PD $= \phi$; then, by well known formulæ,

$$\cos\tfrac{1}{2}\beta = \cos\phi\sin\tfrac{1}{3}\pi = \tfrac{1}{2}\cos\phi\sqrt{3}, \quad \text{therefore } \cos\beta = \tfrac{3}{2}\cos^2\phi - 1.$$

5390. (By Professor LLOYD TANNER, M.A.)—When an equation $F(s, p, q, z, x, y) = 0$ has two independent first integrals, each involving an arbitrary function, it can be reduced to the form $\frac{d^2\zeta}{dx\,dy} = 0$, where ζ is a function of z, x, y.

Solution by the PROPOSER.

It is clear that no first integral of

$$F(s, p, q, z, x, y) = 0 \quad \text{.................................(1)}$$

can involve both p, q. For the x-derivative of such an expression would involve r, s; the y-derivative would involve s, t; and it would be impossible from these to obtain (1), which involves neither r nor t. Say, then, a first integral involves p but not q. We can form (1) by differentiating with respect to y, and in the process we can eliminate an arbitrary function of x. Hence such a first integral may be put into the form

$f(p, z, x, y) = \phi(x)$, and (1) is equivalent to $\frac{df}{dp}s + \frac{df}{dz}q + \frac{df}{dy} = 0$ ...(2).

If there be another first integral, independent of the one above, it must be of the form $\qquad f_1(q, z, x, y) = \psi(y)$;

so that (1) may be written $\frac{df_1}{q}s + \frac{df_1}{z}p + \frac{df_1}{dx} = 0$............................(3).

Suppose, now, (1) is solved for s; then, by (2), it will be linear in q, since f does not involve q. Also, by (3), it will be linear in p, since f_1 does not involve p. Hence, (1) may be written

$$s + Apq + Pp + Qq + Z = 0 \quad \text{..............................(4),}$$

A, P, Q, Z being functions of x, y, z.

On comparing (4) with (2), we get

$$(Ap + Q)\frac{df}{dp} = \frac{df}{dz}, \quad (Pp + Z)\frac{df}{dp} = \frac{df}{dy};$$

similarly $\qquad (AP + P)\frac{df_1}{dq} = \frac{df_1}{dz}, \quad (Qq + Z)\frac{df_1}{dq} = \frac{df_1}{dx}.$

Eliminating f, f_1 from these equations, we find

$$\frac{dA}{dy} = \frac{dP}{dz}, \quad \frac{dA}{dx} = \frac{dQ}{dz}, \quad \frac{dP}{dx} = \frac{dQ}{dy} = AZ + \frac{dZ}{dz} - PQ \text{.........(5),}$$

and these are, as we proceed to show, sufficient to ensure the possibility of the indicated transformation.

In virtue of the first three conditions, $A\,dz + P\,dy + Q\,dx$ is an exact differential, $= dM$ say; and

$$\frac{dM}{dz} = A, \quad \frac{dM}{dy} = P, \quad \frac{dM}{dx} = Q \text{........................(6).}$$

Also we can find a function $f(x, y, z)$ such that

$$\frac{df}{dz} = e^M, \quad \frac{d^2f}{dx\,dy} = Ze^M \quad \text{..........................(7),}$$

for the only condition that this may be possible is

$$\frac{d}{dz}\,Ze^{M} = \frac{d^2e^{M}}{dx\,dy}, \text{ or } \frac{dZ}{dz} + AZ = \frac{dP}{dx} + PQ,$$

which is included in (5). This function f we take as our new dependent variable ζ. We have

$$\frac{d^2\zeta}{dx\,dy} = \frac{d^2f}{dx\,dy} + p\,\frac{d^2f}{dy\,dz} + q\,\frac{d^2f}{dx\,dz} + pq\,\frac{d^2f}{dz^2} + s\,\frac{df}{dz}$$
$$= \{Z + Pp + Qq + Apq + s\}e^{M}, \text{ by (6), (7)}$$
$$= 0, \text{ by (4)}.$$

It may be noticed that (5) form a generalization of Euler's conditions of integrability of (4) when A vanishes.

5033. (By S. Tebay, B.A.)—The roots of a cubic equation are x_1, x_2, x_3. Given $x_1+x_2+x_3 = a$, and $x_1 > x_2 > x_3$; show that the mean form of the equation is $x^3 - ax^2 + \frac{1}{4}a^2 - \frac{1}{60}a^3 = 0.$

Solution by the Proposer; F. C. Mathews, M.A.; *and others.*

We have to calculate the mean values of $x_2x_3 + x_3x_1 + x_1x_2$ and $x_1x_2x_3$. Let $t = x_1+x_2$; then the first of these is $t\,(x_2+x_3) - x_2^2$. Multiply by dx_2, and integrate from $x_2 = x_3$ to $x_2 = \frac{1}{2}t$, and then put $a - x_3$ for t, and we find

$$\tfrac{1}{12}a^3 + \tfrac{1}{4}a^2x_3 - \tfrac{9}{4}ax_3^2 + \tfrac{9}{4}x_3^3.$$

Multiply this by dx_3, and integrate from $x_3 = 0$ to $x_3 = \frac{1}{3}a$, and we get $\frac{1}{48}a^4$. But the number of solutions is $\frac{1}{12}a^2$ (see Quest. 4616). Therefore the average is $\frac{1}{4}a^2$. Again, the mean value of $x_1x_2x_3$ is

$$\frac{12}{a^2}\int_0^{\frac{1}{3}a} x_3\,dx_3 \int_{x_3}^{\frac{1}{2}t} x_2\,dx_2\,(t-x_2) = \frac{1}{a^2}\int_0^{\frac{1}{3}a} x_3\,dx_3\,(a^3 - 3a^2x_3 - 3ax_3^2 + 9x_3^3) = \frac{a^3}{60}.$$

Hence the mean form of the cubic is as given in the Question.

5923. (By D. Edwardes.)—If $x+y+z = 0$, prove that, if x, y, z be any real quantities, the expression $x^3y^3 + y^3z^3 + z^3x^3$ cannot be positive.

Solution by T. Mitcheson, B.A., L.C.P.; J. O'Regan; *and others.*

Since $x^3(y^3+z^3) = x^3(y+z)(y^2+z^2-yz) = -x^4(y^2+z^2-yz)$ is negative, because $y^2+z^2 > yz$ if y, z real; hence $x^3y^3+y^3z^3+z^3x^3$ = a negative.

5612. (By S. Roberts, M.A.)—Prove that (1) the locus of points such that the sum of the squares of their normal distances from a given conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ is constant, is a coaxial and concentric conicoid; (2) if the constant sum is $2(a^2+b^2+c^2)$, the locus becomes a cone; and (3) for a particular value of the constant sum it is identical with the given conicoid, if this is one of a certain family of real similar ellipsoids.

Solution by the Proposer.

To obtain the equation of the locus, we have to find the coefficient of k^{10} in the equation of the parallel where k is the modulus; but since we know that the equation contains x^2, y^2, z^2 only, and linearly, we may suppose $z=0$ in the first instance, and complete by symmetry. Putting $z=0$, we have

$$\left\{(a^2-b^2)^2k^8-2(a^2-b^2)\left[a^4-b^4+(a^2-2b^2)x^2+(2a^2-b^2)y^2\right]k^6+\&c.\right\}$$
$$\times\left\{\frac{c^2}{c^2-a^2}x^2+\frac{c^2}{c^2-b^2}y^2+c^2-k^2\right\}^2=0.$$

And the coefficient of k^{10} is

$$-2(a^2-b^2)\left\{\begin{array}{l}\dfrac{2a^2c^2+2a^2b^2-3b^2c^2-a^4}{c^2-a^2}x^2+\dfrac{3a^2c^2-2b^2c^2-2a^2b^2+b^4}{c^2-b^2}y^2\\ \qquad\qquad+(a^2-b^2)(a^2+b^2+c^2)\end{array}\right\}.$$

Therefore, for the equation of the parallel, we have

$$\begin{array}{l}(a^2-b^2)^2(b^2-c^2)^2(c^2-a^2)^2k^{12}\\ \quad-2k^{10}(a^2-b^2)(b^2-c^2)(c^2-a^2)\\ \\ \qquad+Pk^8+\&c.=0.\end{array}\left\{\begin{array}{l}\quad(2a^2c^2+2a^2b^2-3b^2c^2-a^4)(b^2-c^2)x^2\\ +(2b^2c^2+2b^2a^2-3a^2c^2-b^4)(c^2-a^2)y^2\\ +(2a^2c^2+2b^2c^2-3a^2b^2-c^4)(a^2-b^2)z^2\\ +(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2+b^2+c^2)\end{array}\right\}.$$

Writing A, B, C for a^2, b^2, c^2 respectively, the equation of the locus is

$$\frac{2}{(A-B)(B-C)(C-A)}\left\{\begin{array}{l}\quad(2AC+2AB-3BC-A^2)(B-C)x^2\\ +(2BC+2AB-3AC-B^2)(C-A)y^2\\ +(2AC+2BC-3AB-C^2)(A-B)z^2\end{array}\right\}$$
$$+2(A+B+C)=K^2,$$

where K^2 is the constant sum. If $K^2=2(A+B+C)$ this is a cone.

If the surface is similar to the original conicoid, we have

$$\begin{aligned}(2AC+2AB-3BC-A^2)(B-C)A&=(2BC+2AB-3AC-B^2)(C-A)B\\&=(2AC+2BC-3AB-C^2)(A-B)C,\end{aligned}$$

from which we derive

$$\left.\begin{array}{r}A^2(B+C)+B^2(A+C)+C^2(A+B)-15ABC=0\\ 2(A^2+B^2+C^2)-5(AB+BC+CA)=0\end{array}\right\}\;\cdots\cdots\cdots(a).$$

And if $A_1B_1C_1$ is a common solution, the complete set is obtained by permuting these values.

If we eliminate C from (a), we get an equation of the 6th degree as to

A : B; but, writing p for $\frac{(A+B)^2}{AB}$, this reduces to a cubic; viz.,

$$p^3-21p^2+138p-289=0;$$

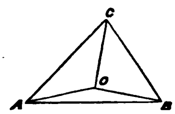
or, writing $q+7$ for p, $\qquad q^3-9q-9=0,$

an equation whose roots are real, lying between -3 and -2, -2 and -1, and $+3$ and $+4$. Hence the roots p lie between 4 and 5, 5 and 6, 10 and 11, and the quadratic $\qquad (A+B)^2-ABp=0$

has real positive roots. Hence the ratios are real and positive, and K^2 can be determined so that the locus shall coincide with the original conicoid. A similar result holds in plane space.

5809. (By the EDITOR.)—If O be any point within a triangle ABC, prove that $OA^n \sin 2A + OB^n \sin 2B + OC^n \sin 2C$ is least when O is the centre of the circumscribed circle.

Solution by E. B. SEITZ.



Let $OA=x$, $OB=y$, $OC=z$, $BC=a$, $CA=b$, $AB=c$, $\angle BOC=\theta$, $\angle COA=\phi$, $\angle AOB=\psi$, and $\angle OCB=\omega$. Then we have

$$x^n \sin 2A + y^n \sin 2B + z^n \sin 2C = \text{a minimum} \quad\ldots(1),$$

$$x^2 = b^2+z^2-2bz\cos(C-\omega) \quad\ldots\ldots\ldots\ldots(2),$$

$$y^2 = a^2+z^2-2az\cos\omega \quad\ldots\ldots\ldots\ldots(3),$$

$$a\sin\omega = y\sin\theta,\quad b\sin(C-\omega)=x\sin\phi \quad\ldots\ldots\ldots\ldots(4,\ 5).$$

Differentiating (1), (2), (3), regarding z as constant, we have

$$x^{n-1}\sin 2A\,dx + y^{n-1}\sin 2B\,dy = 0 \quad\ldots\ldots\ldots\ldots(6),$$

$$x\,dx = -bz\sin(C-\omega)\,d\omega,\quad y\,dy = az\sin\omega\,d\omega \quad\ldots\ldots\ldots(7,\ 8).$$

Substituting these values of dx and dy in (6), we find

$$\frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\phi}{y^{n-1}\sin 2B}.$$

Regarding y as constant, we find, by a similar process,

$$\frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\psi}{z^{n-1}\sin 2C};$$

therefore $$\frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\phi}{y^{n-1}\sin 2B} = \frac{\sin\psi}{z^{n-1}\sin 2C} \quad\ldots\ldots\ldots\ldots(9).$$

Now equations (9) are satisfied by the conditions $x=y=z$, and $\theta=2A$, $\phi=2B$, $\psi=2C$; hence, for a minimum, O must be the centre of the circumscribed circle.

5859. (By J. J. WALKER, M.A.)—If α, β, γ are any three vectors, show that $(T\alpha)^2 S\beta\gamma + S\gamma\alpha S\alpha\beta = S(V\alpha\beta . V\alpha\gamma)$, and verify the result trigonometrically.

Solution by E. ANTHONY, M.A.; EDWARD RUTTER; *and others.*

We have $V\alpha\beta = -V\beta\alpha = -\beta\alpha + S\alpha\beta, \quad V\alpha\gamma = \alpha\gamma - S\alpha\gamma$;

hence $V\alpha\beta . V\alpha\gamma = -\beta\alpha^2\gamma + \beta\alpha S\alpha\gamma + \alpha\gamma S\alpha\beta - S\alpha\beta . S\alpha\gamma$,

and
$$S(V\alpha\beta . V\alpha\gamma) = -\alpha^2 S\beta\gamma + S\beta\alpha . S\alpha\gamma + S\alpha\gamma . S\alpha\beta - S\alpha\beta . S\alpha\gamma$$
$$= -\alpha^2 S\beta\gamma + S\alpha\beta . S\alpha\gamma.$$

If OA, OB, OC are the vectors α, β, γ respectively, and a, b, c the arcs joining the points in which they meet a sphere having O as centre, then we have
$$(T\alpha)^2 S\beta\gamma = -OA^2 . OB . OC \cos a,$$
and
$$S\alpha\beta . S\gamma\alpha = OA^2 . OB . OC \cos b \cos c,$$
so that
$$(T\alpha)^2 S\beta\gamma + S\gamma\alpha . S\alpha\beta = -OA^2 . OB . OC (\cos a - \cos b \cos c)$$
$$= -OA^2 . OB . OC \sin b \sin c \cos A.$$

Now $V\alpha\beta$, $V\alpha\gamma$ are vectors perpendicular to the planes of OA, OB and OA, OC respectively, of lengths OA . OC $\sin b$, OA . OB $\sin c$ respectively. and containing an angle equal to A.

Hence, also, $S(V\alpha\beta . V\alpha\gamma) = -OA^2 . OB . OC \sin b \sin c \cos A.$

5956. (By Professor TOWNSEND, F.R.S.)—A slender uniform circular ring being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere in its space; show that the attraction:—

(*a*) For all positions of the particle, passes in direction through the vertex of the cone which envelopes along the circumference of the ring its sphere of connection with the position of the particle.

(*b*) For all positions of the particle, varies in magnitude directly as the distance from the centre, and inversely as the square of the product of the greatest and least distances from the circumference of the ring.

(*c*) For positions of the particle limited to a definite sphere passing through the circumference of the ring, varies in magnitude directly as the distance from the fixed centre through which it then passes, and inversely as the square of the distance from the plane of the ring.

Solution by the PROPOSER.

Of these properties the first and second, from which the third is an obvious corollary, may be easily proved geometrically, but inferred perhaps more readily from the easily obtained value of the potential V of the mass, for any point in its space, corresponding to the law of force, as follows:—

Denoting by a and by m the radius and the mass of the ring, by x and y the coordinates of the position P of the particle perpendicular respec-

Let the equation of a circle be $(x-\mathrm{X})^2+(y-\mathrm{Y})^2=r^2$.

If θ be eccentric angle of a point of intersection of this circle with ellipse,

$$(a\cos\theta-\mathrm{X})^2+(b\sin\theta-\mathrm{Y})^2=r^2,$$

or

$$c^2\cos^2\theta-2a\mathrm{X}\cos\theta+\&\text{c.}=2\mathrm{Y}\sin^2\theta.$$

Squaring each side, the coefficient of $\cos^3\theta$ is easily seen to be $-\dfrac{4a\mathrm{X}}{c^2}$.

Hence $\quad \frac{1}{4}(a\cos\theta_1+a\cos\theta_2+a\cos\theta_3+a\cos\theta_4)=\dfrac{a^2\mathrm{X}}{c^2}=x_0$, by ($a$).

In the same way we find the coefficient of $\sin^3\theta$ to be $-\dfrac{4b\mathrm{Y}}{b^2-a^2}=\dfrac{4b\mathrm{Y}}{c^2}$.

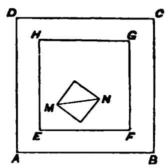
Hence $\quad \frac{1}{4}(b\sin\theta_1+b\sin\theta_2+b\sin\theta_3+b\sin\theta_4)=-\dfrac{b^2\mathrm{Y}}{c^2}=y_0$.

Hence the truth of the problem.

5674. (By E. B. Seitz, M.A.)—Two points are taken at random in the surface of a square; show that the chance that the square, whose diagonal is the line joining the points, lies wholly within the square, is $\frac{2}{3}$.

Solution by the Proposer.

Let ABCD be the given square, M, N the random points. Let MN $=x$, AB $=a$, and θ equal the angle which NM produced makes with AB. Draw the square EFGH, whose sides are at a distance from those of ABCD equal to $\frac{1}{2}x\cos\theta$; then area EFGH $=(a-x\cos\theta)^2$, which represents the number of favourable relative positions of the two points, while their distance and direction from each other are fixed; and, the conditions remaining the same, the whole number of relative positions of the two points is $(a-x\cos\theta)(a-x\sin\theta)$.



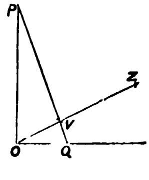
An element of the given square at N is $d\theta\, x\,dx$; also the limits of θ are 0 and $\frac{1}{4}\pi$, and those of x are 0 and $a\sec\theta$. Therefore

$$p=\frac{\displaystyle\int_0^{\frac14\pi}\int_0^{a\sec\theta}(a-x\cos\theta)^2\,d\theta\,x\,dx}{\displaystyle\int_0^{\frac14\pi}\int_0^{a\sec\theta}(a-x\cos\theta)(a-x\sin\theta)\,d\theta\,x\,dx}=\frac{\displaystyle\int_0^{\frac14\pi}\sec^2\theta\,d\theta}{\displaystyle\int_0^{\frac14\pi}(2-\tan\theta)\sec^2\theta\,d\theta}=\frac{2}{3}.$$

5924. (By Simonelli Ruggero.)—On divise une circonférence en douze parties égales, et l'on joint les points de division au centre. Par

5732. (By W. GALLATLY, M.A.)—Find, without differentiation, the envelope of $$x\cos^3\theta + y\sin^3\theta = a.$$

Solution by the Rev. J. L. KITCHIN, M.A.; S. RUGGERO; *and others.*



Writing $x' = a\cos^3\theta$, $y' = a\sin^3\theta$, we have
$$xx' + yy' = a^2;$$
(x, y') being a point on the 4-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; therefore the line is the polar of (x', y') with regard to circle $x^2+y^2=a^2$; therefore the envelope required is the reciprocal polar of the 4-cusped hypocycloid with regard to this circle; therefore, if OV be perpendicular on a tangent to hypocycloid OV, $\text{OZ} = a^2$; Z is point on envelope, and $\text{OV} = a\sin\theta\cos\theta$; therefore $\text{V}\sin\theta\cos\theta = a$ is the envelope.

5864. (By J. YOUNG, B.A.)—Find in what proportion tea at 3*s.* per lb. is mixed with tea at 4*s.* per lb., when by reversing the proportion the quality of the mixture would be improved 10 per cent.

Solution by MORGAN BRIERLEY; H. HARLEY; *and others.*

Let x = the no. of lbs. of tea at 3*s.*; y = the no. of lbs. at 4*s.* When the proportion of tea at 3*s.* to that at 4*s.* is reversed, we have the equation
$$3x + 4y + \frac{3x+4y}{10} = 4x + 3y;\ \text{therefore}\ 2y = x;$$
or 2 lbs. at 4*s.* + 1 lb. at 3*s.* = 11*s.*; and 1 lb. at 4*s.* + 2 lbs. at 3*s.* = 10*s.*, 10 per cent. less.

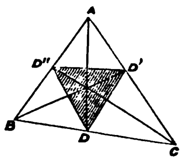
5898. (By R. A. GRAHAM, M.A.)—A conic is drawn through a point P, and the feet of the normals from it to an ellipse. Show that its centre is the centroid of the points in which a circle of any radius meets the ellipses, the centre of the circle being at P.

Solution by D. EDWARDES; G. TURRIFF; *and others.*

Let (X, Y) be the coordinates of P. The feet of normals are the intersections of ellipse with the conic $a^2y\text{X} - b^2\text{Y}x - c^2xy = 0$,
and this evidently passes through X, Y. If (x_0, y_0) are the coordinates of its centre, we easily find $x_0 = \dfrac{a^2\text{X}}{c^2}$, $y_0 = -\dfrac{b^2\text{Y}}{c^2}$ .... (α).

Solution by Professor COCHEZ; D. O'REGAN; *and others.*

Les triangles CDD′, ABC, ayant un angle commun C, sont entre eux comme les côtés qui comprennent cet angle. Donc



$$\frac{CDD'}{ABC} = \frac{CD \cdot CD'}{AC \cdot BC},$$

de même AD′D″ et ABC, BDD″ et ABC donnent respectivement

$$\frac{AD'D''}{ABC} = \frac{AD' \cdot AD''}{AB \cdot AC}, \quad \frac{BDD''}{ABC} = \frac{BD \cdot BD''}{AB \cdot BC}.$$

Si l'on pose $BC = a$, $AC = b$, $AB = c$, et si l'on remarque que D, D′, D″ partagent respectivement les côtés BC, AC et AB en parties proportionelles aux côtés adjacents, on a facilement

$$CD = \frac{ab}{b+c}, \quad CD' = \frac{ab}{a+c}, \quad AD' = \frac{bc}{a+c}, \quad AD'' = \frac{bc}{a+b},$$

$$BD = \frac{ac}{b+c}, \quad BD'' = \frac{ac}{a+b};$$

dès lors

$$\frac{CDD'}{ABC} = \frac{ab}{(b+c)(a+c)}, \quad \frac{BDD''}{ABC} = \frac{ac}{(b+c)(a+b)}, \quad \frac{AD'D''}{ABC} = \frac{bc}{(a+c)(a+b)};$$

d'où $$\frac{\Sigma}{\Delta} = 1 - \frac{ab}{(b+c)(a+c)} - \frac{ac}{(b+c)(a+b)} - \frac{bc}{(a+c)(a+b)};$$

enfin $$\frac{\Sigma}{\Delta} = \frac{2abc}{(a+b)(b+c)(a+c)} = \frac{(a+b)(b+c)(a+c)}{2abc}.$$

[Prof. COCHEZ remarks that his "Δ est l'aire du triangle ABC, Σ l'aire du triangle DD′D;" and he adds furthermore, that "On peut proposer la même question en remplaçant le mot 'bissectrices' par '*hauteurs.*'"]

5680. (By W. J. C. SHARPE, B.A.)—Find (1) the invariant conditions that two conics should have double contact; and (2) the equation to the common tangents in terms of the quantics and their invariants.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

For double contact, the equation to the chords of intersection

$$\Delta S'^3 - \theta S'^2 S + \theta' S' S^2 - \Delta' S^3 = 0 \quad \text{(SALMON's } \textit{Conics}\text{, p. 322)}$$

must have a pair of equal roots denoting the common chord $S' - S$; therefore the above equation must be equivalent to

$$(S'-S)^2(\Delta' S - \Delta S') = 0; \text{ and therefore } \theta = \Delta' + 2\Delta, \; \theta' = 2\Delta' + \Delta,$$

which are equivalent to $\theta - \theta' = \Delta - \Delta'$ and $\theta + \theta' = 3(\Delta + \Delta')$; and the equation to the common tangents is $\Delta S' - \Delta' S = 0$.

5493. (By Professor SYLVESTER, F.R.S.)—If χ_q represent in general the number of linearly independent covariants of the degree q in the variables, and of a given order j in the constants belonging to a binary quantic of the degree i, prove that

$$\chi_0+2\chi_1+3\chi_2+4\chi_3+5\chi_4+\ \dots\dots = \frac{\Pi(i+j)}{\Pi i\,.\,\Pi j}.$$

Note.—Of course χ_0 represents the number of invariants, and the series stops spontaneously after the $(ij+1)^{\text{th}}$ term by all the subsequent χ's vanishing. Of course, when ij is even, all the odd-indiced χ's, and when ij is odd, all the even-indiced χ's, vanish. The theorem as stated comprehends both of these cases in one.

Note by the PROPOSER.

This is a grand theorem which occurs as a corollary to my proof of the *Fundamental Theorem* in Invariants, namely, of the linear independence of a certain general system of derived functions which was established by experience a quarter of a century ago, but had never been demonstrated. I told a pupil at Johns Hopkins University not long ago that, although the fact of this independence was perfectly certain, "it transcended, I thought, the powers of the human understanding to give a proof of it." That proof occurred to me the day before yesterday quite suddenly, like a ray of light entering my brain, and will be sent to CRELLE's *Journal*. It is, I think, the most wonderful kind of proof ever given of an algebraical fact.

5921. (By R. A. ROBERTS, M.A.)—In the equilibrium of a uniform cord stretched over a smooth sphere and repelled according to the law of the inverse cube of the distance by a uniform circular ring lying on the sphere, if the tension at any point be that to infinity under the action of the force, prove that the catenary will be a circle orthogonal to the ring.

Solution by Professor TOWNSEND, F.R.S.

This pretty extension to a sphere of a known property of two uniform orthogonal circles in a plane, noticed I believe first by myself, may be readily reduced to its analogue in the plane, as follows:—

It may be easily shewn, without any difficulty whatever, that the repulsion of the ring, at any point on the sphere, passes in direction through the harmonic pole of its plane with respect to the surface, and varies in magnitude directly as the distance from the pole and inversely as the square of the distance from its plane. Hence, for any circle of the sphere whose plane passes through the pole, and whose circumference consequently intersects at right angles that of the ring, the repulsion varies directly as the distance from that point and inversely as the square of the distance from its polar with respect to the circle. But that is precisely the law of central force requisite to the free equilibrium of a uniform cord in a circular catenary, for which the tension of maintenance

is, as is well known, that to infinity under the action of the force; and therefore, &c.

N.B.—As the ring, in the above, may manifestly be flexible as well as the cord, and both perfectly free in the space of the sphere, as far as the equilibrium of either under the repulsive action of the other for the law of force is concerned, the entire action at every point of each lying in its plane and over the entire circumference of each making equilibrium in its plane; and, as any two circles in a common space which intersect each other twice in the space lie manifestly on a common sphere in the space, it follows consequently from the above, as originally stated by myself for the case of a common plane, that *two uniform circular cords, intersecting twice at right angles in a common space, and repelling each other according to the law of the inverse cube of the distance, will keep each other in free equilibrium in the space.*

5907. (By Professor WOLSTENHOLME, M.A.)—If S_r denote the infinite series $\quad 1^{r-1}+2^{r-1}+\dfrac{3^{r-1}}{2!}+\ldots+\dfrac{(n+1)^{r-1}}{n!}+\ldots$ to ∞,

prove that $\quad \dfrac{S_r}{S_1}=\Delta 0^r+\dfrac{\Delta^2 0^r}{2!}+\dfrac{\Delta^3 0^r}{3!}+\ldots+\dfrac{\Delta^r 0^r}{r!}$, a finite series.

Solution by E. B. ELLIOTT, M.A.; H. STABENOW, M.A.; *and others.*

Here $\quad S_r=0^r+1^r+\dfrac{2^r}{2!}+\dfrac{3^r}{3!}+\ldots$

$$=\epsilon^{E}0^r=\epsilon\,.\,\epsilon^{\Delta}0^r=\epsilon\left\{1+\Delta+\frac{\Delta^2}{2!}+\frac{\Delta^3}{3!}+\ldots\right\}0^r;$$

also $\quad S_1=1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\ldots=\epsilon;$

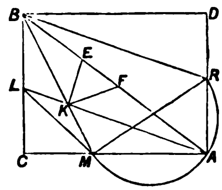
therefore $\quad \dfrac{S_r}{S_1}=\left\{1+\Delta+\dfrac{\Delta^2}{2!}+\dfrac{\Delta^3}{3!}+\ldots\right\}0^r$

$$=\Delta 0^r+\frac{\Delta^2 0^r}{2!}+\frac{\Delta^3 0^r}{3!}+\ldots+\frac{\Delta^r 0^r}{r!},$$

the number of terms being finite as stated, for $\Delta^m 0^r$ vanishes if m be greater than r. [See Question 1653 of the second edition of Prof. WOLSTENHOLME's admirable collection of Mathematical Problems.]

5885. (By the EDITOR.)—Determine a right-angled triangle, having given the two lines bisecting the acute angles, and terminated by the opposite sides; and show that, if the lengths of the bisectors be 40 and 50, the lengths of the sides will be 35·80738, 47·40728, 59·41143.

I. *Solution by* R. Tucker, M.A.



Take a rectangle ACBD; and let AL, BM, the bisectors of A, B, intersect in K; then $\angle$AKB = 135°. Through B draw BR parallel to AL to meet AD in R, then BR = AL. Hence, from data, the triangle BMR is known.

It is well known that

$$2\Delta \text{BMR} + \text{CM} \cdot \text{DR}$$
$$= \text{rect. AB} = 2\Delta\text{BMR} + 2\Delta\text{CLM} \;\ldots\ldots (1).$$

Take BE = BL, AF = AM, then

$$\Delta\text{BKE} = \Delta\text{BKL},\ \Delta\text{AKF} = \Delta\text{AKM},\ \text{and}\ \Delta\text{FKE} = \Delta\text{LKM},$$

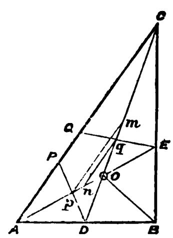
because the angles LKM, FKE are supplementary; therefore quadrangle AMLB = 2ΔAKB; hence, by (1), ΔAKB = ½ΔBMR.

Construction.—Make a triangle BMR, having its sides BM, BR equal to the given bisectors, and the angle MBR equal to half a right angle. On MR draw a semicircle; and construct an hyperbola having BM, BR, for asymptotes, and such that the rectangle under the ordinate and abscissa (parallel to the asymptotes) is half the rectangle under the given bisectors.

Let this hyperbola cut the semicircle in A; join AB, and produce AK, parallel to BR, so that AL = BR, and produce BL, AM to meet in C.

Then ABC will be the triangle required.

II. *Solution by the* Proposer.



1. Let ABC be the triangle; AE $(=2a=40)$, CD $(=2b=50)$ the bisectors, which meet in O; DpP, EqQ perpendiculars to AE, CD respectively; and pm, qn parallels to AC. Then DpP is bisected in p, and EqQ in q; consequently, CD is bisected in m, and AE in n.

And since we have

$$\angle \text{COE} = \tfrac{1}{2}\angle \text{A} + \tfrac{1}{2}\angle \text{C} = 45° = \angle \text{DOA} = \angle \text{CBO},$$

therefore $\quad \text{O}p = \dfrac{\text{OD}}{\sqrt{2}}, \quad$ and $\quad \text{O}q = \dfrac{\text{OE}}{\sqrt{2}}.$

Let O$n = x$, and O$m = y$, then we have

$$\frac{\text{OA}}{\text{OC}} = \frac{\text{O}p}{\text{O}m} = \frac{\text{O}n}{\text{O}q}, \quad \text{or} \quad \frac{a+x}{b+y} = \frac{b-y}{\sqrt{2}\,.\,y} = \frac{\sqrt{2}\,.\,x}{a-x};$$

whence $\dfrac{x}{a} = \dfrac{b-y}{b+y}$; and $\left(\dfrac{y}{b}\right)^3 + \left(\dfrac{y}{b}\right)^2 + \dfrac{2\sqrt{2}\,.\,a-b}{b}\left(\dfrac{y}{b}\right) = 1 \ldots(a).$

Having thus found x and y, we have

$$\text{AC} = \left\{(a+x)^2 + \sqrt{2}\,(a+x)\,(b+y) + (b+y)^2\right\}^{\frac{1}{2}};$$
$$\text{AB} = \frac{\text{AO}^2}{\text{AD}} = \frac{(a+x)^2}{\left\{(a+x)^2 - \sqrt{2}\,(a+x)\,(b-y) + (b-y)^2\right\}^{\frac{1}{2}}}.$$

2. Otherwise:—Put $\angle \text{BAC} = 2\theta$, then we have

$$a\cos\theta\tan 2\theta = b\cos(45° - \theta);$$

or $$\tan^3\theta+\tan^2\theta+\frac{2\sqrt{2}\,.\,a-b}{b}\tan\theta=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$
an equation wherefrom all the parts of the triangle may be readily determined.

3. Or, again, for a third solution, let

$$AE=a=40,\ CD=b=50,\ BC=x,\ AB=y,\ AC=z;$$

then we have $x^2+y^2=z^2$, or $\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

Now, since $AC+CB : CB = AB : BD = \frac{xy}{x+z}$,

therefore $$b^2=x^2+\frac{x^2y^2}{(x+z)^2};$$

therefore $b^2(x+z)^2=x^2\{y^2+(x+z)^2\}=2x^2z(x+z)$, by (1);

whence we obtain $$b^2(x+z)=2x^2z \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

also, by similarity, $$a^2(y+z)=2y^2z \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Equation (2)÷(3) gives $$\frac{b^2}{a^2}\cdot\frac{x+z}{y+z}=\frac{b^2}{a^2}\cdot\frac{\frac{x}{z}+1}{\frac{y}{z}+1}=\frac{x^2}{y^2} \;\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Assume $\frac{x}{z}=\frac{2v}{1+v^2}$, then (1) gives $\frac{y}{z}=\frac{1-v^2}{1+v^2}$, and $\frac{x}{y}=\frac{2v}{1-v^2}$.

Substituting these values in (4), we have

$$\frac{b}{a}\cdot\frac{1+v}{\sqrt{2}}=\frac{2v}{1-v^2},$$

or $$v^3+v^2+\frac{2\sqrt{2}\,.\,a-b}{b}v=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

With the values of a and b given above, equation (γ) becomes

$$v^3+v^2+1{\cdot}26274169v=1,$$

whence we obtain

$$v=\cdot497881647,\quad \frac{x}{z}=\frac{2v}{1+v^2}=\cdot79796006,\quad \frac{y}{z}=\frac{1-v^2}{1+v^2}=\cdot60271033.$$

By (2) and (3), since $a=40$, $b=50$, we have

$$x^2=1250\left(\frac{x}{z}+1\right)=2247{\cdot}45008,\quad y^2=800\left(\frac{y}{z}+1\right)=1282{\cdot}168264;$$

and from these values we obtain, at once,

$$x=47{\cdot}407279=BC,\quad y=35{\cdot}807377=AB,\quad z=59{\cdot}41143=AC.$$

It will be observed that the equations (α), (β), (γ) are the same in the three solutions.

5881. (By Professor DARBOUX.)—Sur un cercle donné O, on prend à volonté un arc ANB, et, sur la corde AB de cet arc, on décrit une demi-

circonférence AMB, puis on fait tourner la figure autour du diamètre perpendiculaire à AB. On demande quelle doit être cette corde pour que la somme des surfaces décrites par les lignes AMB, ANB, soit maximum.

Solution by E. ANTHONY, M.A.; W. J. MACDONALD, M.A.; *and others.*

If C be the middle point of the chord AB, a the radius of the circle, and $OC=y$, we have the surfaces traced out by the semicircle BMA and the arc BNA equal, respectively, to

$$2\pi(a^2-y^2), \text{ and } 2\pi a\int_{-y}^{a} dx \text{ or } 2\pi a(a+y);$$

therefore $\qquad$ sum of the surfaces $= 2\pi(a+y)(2a-y)$.

And this is a maximum when $y=\frac{1}{2}a$; that is, when the chord AB $= a\sqrt{3}$.

5897. (By C. LEUDESDORF, M.A.)—Show that, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$,

$$\left(\frac{a^2-bc}{b+c}+\frac{b^2-ca}{c+a}+\frac{c^2-ab}{a+b}\right)\left(\frac{b+c}{a^2-bc}+\frac{c+a}{b^2-ca}+\frac{a+b}{c^2-ab}\right)=3\frac{a^3+b^3+c^3}{abc}.$$

Solution by the REV. D. THOMAS, M.A.; A. W. SCOTT, M.A.; *and others.*

If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0=\Sigma\frac{1}{a}$, $\quad bc=-a(b+c)$, $\quad a^2-bc=a\,.\,\Sigma a$;

$$\therefore\quad \Sigma\frac{a^2-bc}{b+c}\Sigma\frac{b+c}{a^2-bc}=\Sigma\frac{a}{b+c}\Sigma\frac{b+c}{a}=-\Sigma\frac{a^3}{abc}\left\{\Sigma a\,.\,\Sigma\frac{1}{a}-3\right\}=3\frac{\Sigma a^3}{abc}.$$

5912. (By J. J. WALKER, M.A.)—Prove that $V.V\beta\gamma V\gamma\alpha$ represents the moment of force β, acting at the end of the vector α, about the (unit) axis γ.

Solution by G. TURRIFF, M.A.; W. J. MACDONALD, M.A.; *and others.*

Now $\quad V.V\beta\gamma V\gamma\alpha=\alpha S(V\beta\gamma\,.\,\gamma)-\gamma S(\alpha V\beta\gamma)=-\gamma S\alpha\beta\gamma$;

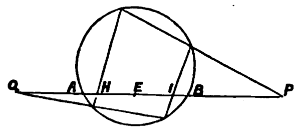
so that the expression represents a vector which represents the moment in direction. Also, if θ be the angle between α and β, and ϕ the angle between their plane and γ, we have (TAIT'S *Quaternions*, p. 64)

$$S\alpha\beta\gamma=-T\alpha T\beta T\gamma\sin\theta\cos\phi=-T\alpha T\beta\sin\theta\cos\phi,$$

which represents the magnitude of the moment; consequently $V.V\beta\gamma V\gamma\alpha$ represents the moment in magnitude and direction.

5634. (By CHRISTINE LADD.)—A secant, bisected in E, meets the successive sides of a quadrilateral inscribed in a circle in the points P, I, Q, H respectively; show that, if EI = EH, EP = EQ.

Solution by the PROPOSER.



If the secant meet the circle in the points A, B, we have, by a well-known theorem,

$$HA \,.\, HB \,.\, IP \,.\, IQ = IA \,.\, IB \,.\, HP \,.\, HQ \ldots (m).$$

But $\qquad HA \,.\, HB = \overline{AE}^2 - \overline{HE}^2,$

$$IP \,.\, IQ = (PE - EI)(QE + EI), \quad IA \,.\, IB = \overline{AE}^2 - \overline{IE}^2,$$

$$HP \,.\, HQ = (PE + HE)(QE - HE).$$

Substituting these values in (m), we have

$$(PE - QE)(\overline{AE}^2 - IE \,.\, HE) = (IE - HE)(\overline{AE}^2 - PE \,.\, QE).$$

If $IE - HE = 0$, then $(PE - QE)(\overline{AE}^2 - IE \,.\, HE) = 0$; but, since $\overline{AE}^2$ is not equal to $\overline{IE}^2$, this must be equivalent to $PE = QE$.

5922. (By R. E. RILEY, B.A.)—Tangents are drawn to an ellipse from points in a concentric circle, radius m; show that the middle points of the chords of contact lie on the curve $r(a^2 \sin^2\theta + b^2 \cos^2\theta) = a^2b^2m^{-1}$.

Solution by C. SHARP, B.A.; R. KNOWLES, L.C.P.; *and others.*

Take the common centre O as origin, and let P (m, ϕ) be a point on the circle; then the equation to a chord of contact of tangents drawn from P is

$$mr(b^2 \cos\theta \cos\phi + a^2 \sin\theta \sin\phi) = a^2b^2;$$

hence, since OP bisects the chord of contact, by putting $\phi = \theta$, we obtain

$$r(b^2 \cos^2\theta + a^2 \sin^2\theta) = a^2b^2m^{-1}, \text{ the required locus.}$$

5790. (By E. W. SYMONS.)—Prove that the equation of the tangent to an involute of $u = 0$, at the point corresponding to (xy), is

$$(X - x)\frac{du}{dy} - (Y - y)\frac{du}{dx} = s\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)^{\frac{1}{2}},$$

s being the length of the arc of u measured from a certain fixed point on the curve to (x, y).

Solution by W. T. CURRAN SHARP, B.A.; *the* PROPOSER; *and others.*

The tangent at any point of the involute is parallel to the normal at the corresponding point of the curve, since both are perpendicular to the tangent to the curve at this point. Hence the equations to the tangent to the involute, and the perpendicular thereupon from (x, y), are

$$(X-x)\frac{du}{dy}-(Y-y)\frac{du}{dx}=p, \quad p\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\}^{-\frac{1}{2}},$$

and the latter must be equal to the length of the tangent to the curve at (x, y), that is, by definition, of involute to S; therefore

$$p = s\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\}^{\frac{1}{2}},$$

and the equation to the tangent to the involute is as in the Question.

5664. (By Professor TAIT, F.R.S.)—Show that the greatest amount of mechanical effect which can be obtained from a system of equal and similar masses (whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures.

Solution by J. J. WALKER, M.A.

The deduction of this particular case from the equations applicable to a more general system has been made by Professor TAIT himself in his *Thermo lynamics*, 2nd ed. § 181; but the object in proposing it as above appearing to be to elicit a simple proof specially applicable to this case, the following may be given.

Using Prof. TAIT's notation, let m be the common mass, t, t' the absolute temperatures, c the specific heat, while T is the absolute temperature to which the system is brought, so that all the heat lost is converted into work. Then $W = Jcm[t-T-(T-t')] = Jcm(t+t'-2T)$; but $\frac{t}{T}=\frac{T}{t'}$, from the definition of absolute temperature, or $T = (tt')^{\frac{1}{2}}$.

Hence finally $\qquad W = 2Jcm\left[\frac{1}{2}(t+t')-(tt')^{\frac{1}{2}}\right].$

5545. (By J. C. MALET, M.A.)—Prove that the locus of the centre of a circle which cuts orthogonally a given circle, and touches a given conic, is a curve of the sixth order, which has six cusps that lie all on another conic; and hence prove that the six cusps of the first negative pedal of a central conic with respect to any point all lie on another conic.

· *Solution by* J. W. SHARPE, M.A.; E. RUTTER; *and others.*

Take the equation to the conic under the form $\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1=0$, and the equation to the circle under the form

$$\xi^2+\eta^2-2u\xi-2v\eta-w^2=0.$$

Then the condition that this circle be cut orthogonally by the variable circle $(\xi-x)^2+(\eta-y)^2-r^2=0$ is

$$2ux+2vy-x^2-y^2+r^2+w^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Calculating the invariants of the conic and the variable circle, we get

$$\Delta=-\frac{1}{a^2b^2},\quad \Delta'=-r^2,\quad \Theta=\frac{x^2+y^2-r^2-a^2-b^2}{a^2b^2},$$

$$\Theta'=\frac{b^2x^2+a^2y^2-a^2b^2-r^2(a^2+b^2)}{a^2b^2}.$$

We shall express the condition that the variable circle touch the conic; and, eliminating r^2 between the resulting equation and the equation (a) above, we shall obtain the equation to the locus of the point (x, y), which is the centre of the variable circle. Now put

$$L=2ux+2vy+w^2-a^2-b^2,\quad C=r^2 \text{ or } x^2+y^2-L-a^2-b^2,$$

$$\Sigma=b^2x^2+a^2y^2-a^2b^2-(a^2+b^2)\,C;$$

so that $C=0$ is the equation to the given circle; and $\Sigma=0$ is a conic passing through the four points common to the given circle and conic; and $L=0$ is the equation to a straight line. Then

$$\Delta=-\frac{1}{a^2b^2},\quad \Delta'=-C,\quad \Theta=\frac{L}{a^2b^2},\quad \Theta'=\frac{\Sigma}{a^2b^2}.$$

Now the condition that the variable circle should touch the conic is

$$(\Theta\Theta'-9\Delta\Delta')^2=4(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta),$$

which becomes upon substitution

$$(L\Sigma-9a^2b^2C)^2=4(L^2+3\Sigma)(\Sigma^2+3a^2b^2CL).$$

This equation remains of precisely the same form, whatever be the form in which we assume the equation to the given conic. For the coefficients of the conic enter into L under the form a^2+b^2, and into the Σ under the forms a^2+b^2 and a^2b^2; and these are both immediately expressible in terms of the two invariants of the given conic. Hence we have obtained the general equation to the locus of the centre of the variable circle.

The form of the equation shows it to be of the sixth degree. Also it touches the conic $L^2+3\Sigma=0$ in the six points in which the conic is intersected by the cubic $L\Sigma-9a^2b^2C=0$, and touches the quartic $\Sigma^2+3a^2b^2CL=0$ in the twelve points where it is cut by this cubic. Now, if we eliminate C between the last two equations, we get the complex $\Sigma(L^2+3\Sigma)=0$; hence the last twelve points are the intersections of the cubic with the two conics $\Sigma=0$, and $L^2+3\Sigma=0$. Hence it appears that the six points of intersection of the cubic with the latter conic are points where the locus touches both the conic and the quartic; hence these six points are double points of some kind or other. We shall now show that they are cusps.

The equation to the locus becomes, when expanded,

$$L^2\Sigma^2+18\alpha^4L\Sigma C-27\alpha^8C^2+4\alpha^4L^3C+4\Sigma^3=0,$$

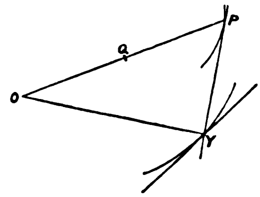
where we have put α^4 for a^2b^2.

Now $$(L^2+3\Sigma)^3=L^6+9L^4\Sigma+27L^2\Sigma^2+27\Sigma^3.$$

Whence it immediately appears that the equation to the locus required may be written in the form $(2L^3+9L\Sigma-27a^4C)^2-4(L^2+3\Sigma)^3=0$, which shows that the six points, in which the cubic $2L^3+9L\Sigma-27a^4C=0$ cuts the conic $L^2+3\Sigma=0$, are cusps. That these are the six double points above discussed becomes evident upon combining these two equations.

The tangents at the cusps are of course the tangents at those points to the cubic.

For the second part of the problem let the given circle reduce to a point circle. The variable circle must then pass through this point, because it cuts the given circle in all cases orthogonally.



Let O be this given point; P the point on the negative pedal which corresponds to the point Y on the conic, so that PY is the tangent to the pedal at P. Then the tangent at Y to the conic and that to the pedal at P are equally inclined to OY and OP respectively; hence the circle described about the triangle OPY touches the conic at Y; also, OYP being a right angle, its centre is Q, the middle point of OP; hence the locus P is similar to the locus of Q. But by the above investigation the latter has six cusps lying upon a conic; therefore so also has the locus of P. Hence the required proposition is proved.

5892. (By the Rev. W. A. Whitworth, M.A.)—If C^n_r denote the number of combinations of n things r together, prove that

$$\Sigma_{\alpha=0}^{\alpha=a}\Sigma_{\beta=0}^{\beta=b}\Sigma_{\gamma=0}^{\gamma=c}\dots\Sigma_{\mu=0}^{\mu=m}\frac{C^a_\alpha C^b_\beta C^c_\gamma\dots C^m_\mu}{C^{a+b+c+\dots+m}_{\alpha+\beta+\gamma+\dots+\mu}}=1+a+b+c+\dots+m.$$

5913. (By the Rev. W. A. Whitworth, M.A.)—If C^n_r denote the number of combinations of n things r together, prove that, whatever be the value of ν,

$$\Sigma_{\alpha=0}^{\alpha=a}\Sigma_{\beta=0}^{\beta=b}\Sigma_{\gamma=0}^{\gamma=c}\dots\Sigma_{\mu=0}^{\mu=m}\frac{C^a_\alpha C^b_\beta C^c_\gamma\dots C^m_\mu C^n_\nu}{C^{a+b+c+\dots+m+n}_{\alpha+\beta+\gamma+\dots+\mu+\nu}}=\frac{1+a+b+c+\dots+m+n}{1+n}$$

Solution of both Questions by the Proposer.

Consider a parallelepiped whose edges are m, n, r, and divide it by lines parallel to its edges into mnr parallelepipeds, all of whose edges are unity.

Let it be required to go from one corner to the opposite corner of the great parallelepiped by traversing edges of the little parallelepipeds without retrogression.

The number of ways is evidently $\frac{(m+n+r)!}{m!\,n!\,r!}$, for which we may conveniently write $C_{m,n,r}$.

Classify these routes accordi...
from the plane $Z = z$ to the ...
as origin and the three edges ...
The number of routes pa... $C_{x,y}$...
$(x, y, z+1)$ is

Give all possible values to x...
routes, therefore

$$\Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} C_{x,y} \ldots$$

or $$1 = \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{C_{x,y,z} C_{\ldots}}{C_{\ldots}}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{(x+y+z)! \ldots}{x!\,y!\,z!\,(\ldots}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{\frac{m!}{x!\,(m-x)!} \ldots}{(x+y+z)! \ldots}$$

Now write a, β, γ for $x, y, \ldots$
equation becomes $\frac{a+b+c+1}{c+1} = \ldots$

which is a particular case of Que...

Now imagine space of N dimen...
as before; then, if the N coordinat...
constants $a, b, c \ldots m, n$, our resu...

$$\frac{a+b+c+\ldots+m+n+1}{n+1} = \Sigma \ldots$$

which is the theorem of Question ...

The sum $\frac{a+b+c+\ldots+m+n+1}{n+1}$. Gi...
... for all values of r. Then we get
... and add.

$$a+b+c+\ldots+m+n+1 = \Sigma_{a=0}^{a=a} \Sigma_{\beta=0}^{\beta=b} \ldots$$

which is the theorem of Question 5592, ...

(By H. L. Orchard, L.C.P.)—T...
... from water, the other from a ...
... superficial tensions per linear ...
... and the radii be ⅔ inch and 1¼ in...
... the excess of the total internal ...

Classify these routes according to the ways in which they severally pass from the plane $Z = z$ to the plane $Z = z+1$, regarding the initial point as origin and the three edges through this point as axes of coordinates.

The number of routes passing from the point (x, y, z) to the point $(x, y, z+1)$ is $\quad C_{x,y,z} \cdot C_{m-x,\, n-y,\, r-z-1}$.

Give all possible values to x and y and the summation must give all the routes, therefore

$$\Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} C_{x,y,z} \cdot C_{m-x,\, n-y,\, r-z-1} = C_{m,n,r},$$

$$\text{or} \quad 1 = \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{C_{x,y,z}\, C_{m-x,\, n-y,\, r-z-1}}{C_{m,n,r}}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{(x+y+z)!\,(m+n+r-x-y-z-1)!\,m!\,n!\,r!}{x!\,y!\,z!\,(m-x)!\,(n-y)!\,(r-z-1)!\,(m+n+r)!}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{\dfrac{m!}{x!\,(m-x)!}\,\dfrac{n!}{y!\,(n-y)!}\,\dfrac{(r-1)!}{z!\,(r-z-1)!}}{\dfrac{(m+n+r-1)!}{(x+y+z)!\,(m+n+r-x-y-z-1)!}} \cdot \frac{r}{m+n+r}$$

Now write α, β, γ for x, y, z, and a, b, c for $m, n, r-1$. Then our equation becomes $\dfrac{a+b+c+1}{c+1} = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \dfrac{C_\alpha^a\, C_\beta^b\, C_\gamma^c}{C_{\alpha+\beta+\gamma}^{a+b+c}}$,

which is a particular case of Question 5943.

Now imagine space of N dimensions and make the same construction as before; then, if the N coordinates be $\alpha, \beta, \gamma \ldots \mu, \nu$, and the corresponding constants $a, b, c \ldots m, n$, our result becomes

$$\frac{a+b+c+\ldots+m+n+1}{n+1} = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \ldots \Sigma_{\mu=0}^{\mu=m} \frac{C_\alpha^a\, C_\beta^b\, C_\gamma^c \ldots C_\mu^m\, C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}},$$

which is the theorem of Question 5943.

The sum $\dfrac{a+b+c+\ldots+m+n+1}{n+1}$ is independent of ν and will therefore be the same for all values of ν. Give ν all the values $0, 1, 2, \ldots n$ in succession and add. Then we get

$$a+b+c+\ldots+m+n+1 = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \ldots \Sigma_{\mu=0}^{\mu=m} \Sigma_{\nu=0}^{\nu=n} \frac{C_\alpha^a\, C_\beta^b\, C_\gamma^c \ldots C_\mu^m\, C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}},$$

which is the theorem of Question 5892, with n written for m.

5814. (By H. L. ORCHARD, L.C.P.)—Two spherical soap-bubbles are blown, the one from water, the other from a mixture of water and alcohol. If the unit superficial tensions per linear inch be 1 grain and $\frac{7}{12}$ grain respectively, and the radii be $\frac{7}{8}$ inch and $1\frac{1}{3}$ inch respectively, compare, in the two cases, the excess of the total internal pressure over the external pressure.

5944. (By H. L. ORCHARD, L.C.P., &c.)—A spherical soap-bubble is electrified in such a manner that the internal pressure p remains constant. Show that, if ρ_n, ρ_{n+1}, respectively, are the electrical densities when its volume has become n and $n+1$ times its initial volume, we must have the relation $\rho_{n+1} : \rho_n = n : (n^2-1)^{\frac{1}{2}}$.

Solution of both Questions by J. J. WALKER, M.A.

The solution of the former of these questions is obvious, as from the well-known expression $p = \dfrac{2T}{r}$, giving for the total pressure $Sp = 8\pi Tr$, whenever, as in the example proposed, the tensions are inversely as the diameters, the total pressures will be equal.

The second, though not practically verifiable, is ingenious; and such questions have an educational value. Let a be the initial radius, then the work done in enlarging the sphere to radius r by a uniform pressure p,

will be equal to $$4\pi p \int_a r^2\, dr = \tfrac{4}{3}\pi p\,(r^3-a^3),$$

which, when $r^3 = na^3$, reduces to $\frac{4}{3}\pi p\,(n-1)\,a^3$, and when $r^3 = (n+1)\,a^3$, to $\frac{4}{3}\pi p\, na^3$. Equating these to the potential energies of the electrifications, *i.e.*, to $\dfrac{Q^2}{2r}$, half the product of charge and potential, according to Helmholtz's formula, or $8\pi^2\rho^2 r^3$, in each case—ρ being the electrical density—there results

$$(n+1)\,\rho_{n+1}^2 : n\rho_n^2 = n : n-1 \quad \text{or} \quad \rho_{n+1}^2 : \rho_n^2 = n^2 : n^2-1.$$

5742. (By FREDERICK PURSER, M.A.)—A conic U touches a line L and a conic V, and has double contact with another conic W; show that the chord of contact of U and W envelopes a binodal quartic of which the points of intersection of L and W are the nodes.

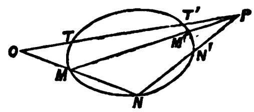
Solution by the PROPOSER.

1. The bicircular quartic is generated as the envelope of a circle whose centre lies on a conic F and which cuts a circle J orthogonally. Hence follows readily the following generation of the curve as a locus. Let a line parallel to the polar with respect to a circle of any point P, and bisecting the interval between P and the polar, be termed the quasi-polar of P with respect to the circle; then the bicircular quartic is the locus of points whose quasi-polars with respect to the circle J touch the conic F.

If now the two circular points at infinity be replaced by any two points M, N, the circle J becomes a conic through MN, and the quasi-polar of any point P is now the chord joining the other intersections with J of the

lines PM, PN; the binodal quartic being then the locus of points whose quasi-polars to J touch F.

2. Now it may be shown as follows, that the quasi-polars with respect to J of points on a given right line envelope a conic having double contact with J and touching the line MN.



For, let M′, N′, T, T′, be the conic J; T, T′ the given line; P any point on it; M′N′ its quasi-polar. Then, as P moves along TT′, the points M′, N′ trace out homographic systems, and therefore M′N′ envelopes a conic U having double contact with J. When P is at T, T′, the chord M′N′ becomes the tangent to J at T, T′, showing that the given line TT′ is the chord of contact of J with the envelope conic. Lastly, let P be at O the intersections of MN, TT′, then M′N′ coincides with MN. Hence MN is a line of the enveloping system, or the envelope conic U touches MN.

3. Let it now be required to find the points in which a given line TT′ meets the binodal quartic. From what has been proved, it appears that the quasi-polars of these points are the four common tangents to the conic F and the envelope conic U. The four corresponding poles on TT′ are then the points required. Let now the envelope conic U touch F; then two of these points will coincide, or the line TT′ will touch the binodal quartic. Hence any tangent TT′ to the binodal quartic is the chord of contact with J of a conic U touching the line MN, the conic F, and having double contact with J.

4. I would add that the same method leads to the more general theorem, "A conic U touches a line L, has double contact with a conic W, and touches a given curve of the n^{th} class. The envelope of the chord of contact of U and W is a curve of degree $2n$, having the two points of intersections of L and W as multiple points of order n.

5813. (By W. R. Roberts, M.A.)—From any point on an inflexional tangent to a cubic can be drawn four tangents to the curve. Show that there are six points, two on each inflexional tangent lying on a conic, from which the tangents form an harmonic pencil.

Solution by J. J. Walker, M.A.

It seems as if this theorem had been obtained by transformation for a nodal cubic, and it had been inferred that it must be true for non-singular cubics also; but, on testing this generalisation, it appears not to be justified.

Let a cubic referred to the three inflexion tangents be

$$(x+y+z)^3 + 3mxyz = 0;$$

then, if from $(x'y')$, a point on $z=0$, tangents are drawn to the curve, I find

for the condition that they should form a harmonic pencil, if $w' = x' + y'$,

$$27w'^{12} + 54mw'^{10}x'y' + 45m^2w'^8x'^2y'^2 - 180m^2w'^6x'^3y'^3$$
$$-15(12m^3+m^4)w'^4x'^4y'^4 - 6(10m^4+m^5)w'^2x'^5y'^5$$
$$-(24m^4+12m^5+m^6)x'^6y'^6 = 0,$$

which, on examination, will not, I think, be found to have a quadratic factor; but if the condition for the curve being nodal be introduced—viz., $m = -9$,—then the above condition becomes $27(w'^2 - 3x'y')^6 = 0$, and there are, accordingly, two points on each tangent possessing the property. The six points evidently lie on the conic

$$x^2 + y^2 + z^2 - yz - zx - xy = 0.$$

I have checked the general result above by calculating it independently for the ranges made on the lines $y = 0$ and $x + y + z = 0$; and its reducing, as it does, for $m = -9$ is a further test of its correctness.

5888. (By R. Tucker, M.A.)—D, E, F are the points of contact of an inscribed circle with the sides. Circles are inscribed in AFE, BFD, CDE, and their centres are joined. The triangle A′B′C′ thus formed is treated in the same way as the triangle ABC, and so on: prove that the triangle $A_\infty B_\infty C_\infty$ is equilateral.

Solution by J. O'Regan; *the* Proposer; *and others.*

It is readily seen that A′, B′, C′ are the middle points of the arcs EA′F, FB′D, DC′F. Hence $\angle A'$, which stands upon B′DC′, is half angle on FDE, *i.e.*, $= 45° + \frac{1}{4}A$, therefore $A' - B' = \frac{1}{4}(A - B)$; and similarly

$$A_n - B_n = \frac{A-B}{4^n}, \text{ and } A_\infty = B_\infty = C_\infty,$$

which proves the property.

5937. (By Septimus Tebay, B.A.)—Show that, to the first order of approximation, the equation to the section of a comet (consisting of a small spherical nucleus, surrounded by light nebulous matter) by the plane of its orbit is $r = a - b\cos\theta + c\cos 2\theta$; the equilibrium being statical,—an assumption suggested by the general apparition of all comets.

Solution by the Proposer.

Let a be the radius of the undisturbed mass of the comet, μ the mass of the nucleus, m the mass of the sun, d the distance between their centres, r the distance of any point (xyz) of the nebula, r' its distance from the

$$\frac{dp}{d\theta} = -\,\mathrm{X}\,(y^2+z^2)^{\frac{1}{2}} + \frac{\mathrm{Y}\,xy + \mathrm{Z}\,xz}{(y^2+z^2)^{\frac{1}{2}}} - \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + r^2 \sin\theta\cos\theta\left(n+\frac{d\phi}{dt}\right)^2 \quad \text{.........(B)},$$

$$\frac{dp}{d\phi} = \mathrm{Z}y - \mathrm{Y}z - 2r\frac{dr}{dt}\sin^2\theta\left(n+\frac{d\phi}{dt}\right) - 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}\left(n+\frac{d\phi}{dt}\right) - r^2\sin^2\theta\frac{d^2\phi}{dt^2} \quad \text{.........(C)};$$

p being the pressure at any point of the ocean; r, θ, ϕ the polar coordinates, viz., radius-vector, north polar distance, and longitude, of any point of the ocean; n the angular velocity of the rotating nucleus; x, y, z the coordinates of any point of the ocean referred to three rectangular axes fixed in space, x being the axis of rotation; and X, Y, Z the components of the disturbing forces, resolved parallel to the axes of x, y, z.

These important equations are usually deduced from the differential equations of motion, referred to fixed rectangular coordinate axes, by means of transformation of coordinates, by a process which will be admitted, by any one who has honestly gone through it, to be hideously repulsive.

I say, that the exact equivalents of equations (A), (B), (C) may be written down at sight, and without calculation, from elementary geometrical and mechanical principles. Required to show how this may be done.

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting by ρ the projection of r on the plane of yz, so that

$$\rho^2 = y^2+z^2 = r^2\sin^2\theta,$$

and supposing, as is done in the equations, the density of the water to be unity, we have, by the general equations of fluid motion,

$$\frac{dp}{dx} = \mathrm{X} - \frac{d^2x}{dt^2},\quad \frac{dp}{dy} = \mathrm{Y} - \frac{d^2y}{dt^2},\quad \frac{dp}{dz} = \mathrm{Z} - \frac{d^2z}{dt^2},$$

from which, by resolution along [illegible] three rectan[illegible]tions of r, θ, and ϕ, we get at once

$$\frac{dp}{dr} = \frac{1}{r}[\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z] - [illegible] + y\frac{d^2y}{dt^2} [illegible]$$

$$= \frac{1}{r}[\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z] [illegible] + \frac{dr^2}{dt^2} [illegible]$$

$$= \frac{1}{r}[\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z] - \frac{1}{2r}[illegible] \frac{1}{r}\left[\frac{dr^2}{d}[illegible]\right.$$

$$\frac{dp}{d\theta} = -[illegible] + \mathrm{Z}[illegible]$$

$$= [illegible] + \mathrm{Z}[illegible]$$

$$= [illegible]$$

$$\frac{dp}{d\phi} [illegible]$$

which [illegible]

good example of the very subtle and obscure nature of many quesions on Probability. "Is the reasoning wrong," he adds, "or is it right; and consequently is the paradox to be accepted?" We have a quantity (B) which is certainly likely to be greater than £5, the odds being 4 : 3 in its favour (in fact, the *expectation*, for the cheque B, is greater than £5), and yet there is another unknown A, which is more likely to exceed B, than it is to exceed £5.

2. There is another question which, though of no unfrequent occurrence, and of great importance as lying at the root of a large number of questions in Probability, seems to require more investigation than it has hitherto received. If a bag contains a number of balls about which nothing is known, except that they are black or white, or—what is the same thing—if an event occurs a given number of times, about which we know nothing, except that it can only turn out in two ways, it is commonly assumed that every possible partition of the number into black and white is *à priori* equally likely. Now, if this is the case, say for two bags containing each 10 balls, and if they are all thrown into one bag, it is easy to show that it cannot be true for the 20 balls in this bag; yet how can our knowledge about these 20 balls be affected in any way by their having been originally in two sets? But, if the hypothesis be correct, it must be. Reducing the question to the simplest possible case, let two bags contain 1 ball each; if they are thrown together, the probabilities for the three possible partitions (0, 2), (1, 1), (2, 0) are as 1 : 2 : 1, instead of as 1 : 1 : 1, which the hypothesis gives.

SECOND NOTE ON SOME QUESTIONS IN PROBABILITY. *By the* EDITOR.

1. In reply to the EDITOR's former *Note* hereon (see above) Mr. WHITWORTH makes the following observations:—

(α). "Professor CROFTON's paradox (Question 5853) does not seem to me very paradoxical. It is easy to see, without calculation, that there is more chance of exceeding a fluctuating quantity than a constant quantity, even though the average value of the fluctuating quantity exceed the constant; for example, if the faces of a die be marked 11, 11, 11, 11, 11, 5, and the faces of another die be marked 7, 7, 7, 7, 7, 7, a person throwing with a *common* die has a chance ($=\frac{1}{36}$) of exceeding the first, and he has *no* chance of exceeding the second. In other words, his chance of exceeding 7 is less than his chance of exceeding a quantity which is "likely" to be 11, and which on an average is 10.

(β). "I cannot admit that, if a bag contains a number of balls, about which nothing is known except that they are black or white, *it is commonly assumed* that every possible partition of the *number* into black and white is *à priori* equally likely. What is (rightly) assumed is, that every possible partition of the *things* (treated as "different") is *à priori* equally likely. The chance that among the n balls there are exactly r black ones is therefore $\dfrac{n!}{r!\,\overline{n-r}!}\left(\dfrac{1}{2}\right)^n$, and not $\dfrac{1}{n+1}$ (as the false assumption would give).* But, of course, the assumption which the EDITOR reprobates would be correct in such a case as the following:—A man is to paint n balls, some black and some white. He inquires how many are to be painted black, and the answer (r) is given at random. These balls are then put into a bag, &c."

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* Mr. WOOLHOUSE states that, in his opinion, "this is undoubtedly the correct view."

$r \sin\theta \,.\, (\dot{\phi}+n)^2$ directed towards that axis, together with a motion about OQ as axis giving an acceleration $r\dot{\theta}^2$ directed towards the origin.

The acceleration of the fluid element relatively to this point is, in the same direction, $\ddot{r}$; so that the total acceleration is

$$\ddot{r}-r\dot{\theta}^2-r\sin^2\theta\,(n+\dot{\phi})^2.$$

The resolved part of the disturbing forces is

$$\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right)dv,$$

where dv is the volume of the polar element, and therefore denotes the mass of the fluid contained in the element.

The fluid pressures which take effect in this direction are

$p \times$ (area of that face of the element which passes through P and is at right angles to OP)

$$-\left(p+\frac{dp}{dr}dr\right)\times(\text{area of the opposite face}) = -\frac{dp}{dr}dv;$$

hence, by D'Alembert's principle, reversing the effective forces, and writing down the equation of motion as one of equilibrium; or, by simply equating the motion produced and the forces producing it, we obtain

$$\frac{dp}{dr}=X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}-\ddot{r}+r\dot{\theta}^2+r\sin^2\theta\,(n+\dot{\phi})^2,$$

the equivalent of (A).

(ii.) To obtain equation (B), consider the moment of momentum about the line OQ, considering OQ to be a line fixed in space with which the line at right angles to the plane xOP momentarily coincides. Let ρ denote the projection of OP upon the plane yOz, so that $\rho^2=y^2+z^2$.

The moment of momentum in the plane xOP is $r^2\dot{\theta}\,.\,dv$; its rate of change is $\frac{d}{dt}(r^2\dot{\theta})\times dv$; considering as positive that motion which tends to increase the angle θ. The acceleration in the plane at right angles to Ox is $-r\sin\theta\,(n+\dot{\phi})^2$, giving a moment of momentum

$$-r\sin\theta\,(n+\dot{\phi})^2\times r\cos\theta\times dv = -r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2\times dv;$$

hence moment of effective forces is

$$\left\{\frac{d}{dt}(r^2\dot{\theta})-r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2\right\}dv;$$

moment of disturbing forces is

$$\left\{-X\rho+Y\frac{y}{\rho}\times x+Z\frac{z}{\rho}\times x\right\}dv;$$

moment of fluid pressures is

$$-\frac{dp}{d\theta}\times r\sin\theta\,d\phi\,.\,dr\times r = -\frac{dp}{d\theta}dv:$$

hence

$$\frac{dp}{d}=-X(y^2+z^2)^{\frac{1}{2}}+\frac{Yxy+Zxz}{(y^2+z^2)^{\frac{1}{2}}}-\frac{d}{dt}(r^2\dot{\theta})+r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2 \ldots \text{(B)}.$$

(iii.) For equation (C) takes moments in the same way about the axis of rotation Ox. The moment of momentum of the effective forces is clearly

$$r \sin\theta\, (n+\dot{\phi}) \times r \sin\theta \times dv,$$

of which the rate of change is

$$\frac{d}{dt}\left\{r^2 \sin^2\theta\, (n+\dot{\phi})\right\} \times dv.$$

The moment of the disturbing forces is $(Zy - Yz)\, dv$, and that of the fluid pressures is ultimately

$$-\frac{dp}{d\phi}\, d\phi \times dr\,.\, r\, d\theta\,.\, r \sin\theta = -\frac{dp}{d\phi}\, dv;$$

hence $\dfrac{dp}{d\phi} = Zy - Yz - \dfrac{d}{dt}\left\{r^2 \sin^2\theta\, (n+\dot{\phi})\right\}$, the equivalent of (C).

This method of obtaining the equations appears to satisfy Dr. HAUGHTON's requirements, being perfectly direct, and having no reference, as in the case of Prof. TOWNSEND's method, to transformation of coordinates. Each term in the equations as given above can be written down from first principles, without the necessity of writing down any preliminary calculations.

IV. *Solution by* Professor G. JOHNSTONE ALLMAN, M.A.

The following solution is extracted from the *Report* made by a Mathematical Committee, on January 30th, 1847, on a paper entitled "On the Polar Equations of Dynamics and Hydrostatics," and published in Vol. III. of the *Transactions* of the Dublin University Philosophical Society for the 4th and 5th sessions, ending November, 1847.

"Mr. ALLMAN commenced by stating a principle of considerable importance in many physical investigations, viz.,

The effective moment round any axis

$$= \frac{d}{dt}\left\{\text{Areolar velocity round that axis}\right\};$$

and thought it worth noticing that the truth of this principle may be established by a method similar to that used by Newton in his first Proposition. In the first place, let the motion of a single body, submitted to the action of given forces and moving freely, be considered.

Let the components of the given forces parallel to three fixed axes be X, Y, Z; let r be the radius vector of the centre of gravity of the body, ψ the angle between the radius and the plane of XY, and ϕ the angle which the plane of the radius vector and the axis of Z makes with the plane of ZX.

About the origin as centre and with the radius r let a sphere be described; let the motion of the body be resolved into three rectangular directions, one in the radius r, a second in the meridian plane Zr, and a third in the plane of the parallel circle; then the centrifugal forces, $\left(r\dfrac{d\psi^2}{dt^2}\right.$ acting out from the centre of the sphere, and $r\cos\psi\,\dfrac{d\phi^2}{dt^2}$ acting out from the centre of the parallel$\Big)$, arising from the inertia of the moving body, must be introduced in order to reduce the question to Statics.

Since the body moves freely, the effective force in the radius vector is equal to the sum of the applied forces in that direction, or,

$$(X\cos\phi + Y\sin\phi)\cos\psi + Z\sin\psi + r\frac{d\psi^2}{dt^2} + r\cos^2\psi\frac{d\phi^2}{dt^2} - \frac{d^2r}{dt^2} = 0.$$

Also, the effective moment round the axis of Z is equal to the sum of the applied moments round that axis, or

$$(Y\cos\phi - X\sin\phi)\,r\cos\psi - \frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\} = 0.$$

Again, the effective moment round an axis perpendicular to the plane of the meridian is equal to the sum of the applied moments round that axis, or

$$Zr\cos\psi - (X\cos\phi + Y\sin\phi)\,r\sin\psi - r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2} - \frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\} = 0.$$

Mr. ALLMAN next considered the case of a system of bodies regarded as material particles, mutually attracting each other, submitted to the action of forces directed to fixed centres and moving freely; and proceeded to form the general equation arising from the joint application of D'Alembert's principle, and the principle of virtual velocities, where the polar coordinates r, ϕ, ψ, r', ϕ', ψ', &c. of the different bodies m, m', &c., are taken as the variables.

By means of an important observation of LAGRANGE, *Mécanique Analytique*, Tome I., page 51, that 'if in the general formula of virtual velocities one of the independent variables be the angle of rotation around any axis, then the coefficient of the differential of this angle will be the sum of all the moments relative to this axis,' the general equation

$$\begin{aligned}
&Sm\left[(X\cos\phi + Y\sin\phi)\cos\psi + Z\sin\psi + r\frac{d\psi^2}{dt^2} + r\cos^2\psi\frac{d\phi^2}{dt^2} - \frac{d^2r}{dt^2}\right]\delta r\\
&+ Sm\left[(Y\cos\phi - X\sin\phi)\,r\cos\psi - \frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\}\right]\delta\phi\\
&+ Sm\Bigg[Zr\cos\psi - (X\cos\phi + Y\sin\phi)\,r\sin\psi\\
&\qquad - r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2} - \frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\}\Bigg]\delta\psi = 0^*
\end{aligned}$$

is established in a manner as direct and immediate as the equation

$$Sm\left\{\left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z\right\} = 0.$$

In the case of the motion of fluids, the difference between the sum of the applied forces and the effective forces (also the difference between the sum of the applied moments and the effective moment) must equilibrate by means of the pressure; now the force in the radius vector arising from the pressure is $\frac{dp}{dr}$, the moment of the pressure round the axis of Z is

$$r\cos\psi\frac{dp}{r\cos\psi\,d\phi} = \frac{dp}{d\phi},$$

* This equation is deduced by LAGRANGE from the principle of least action, and is also an elegant consequence of the transformation of the general formula of dynamics given by him in the fourth section of the second part of the *Mécanique*.

and the moment of the pressure round an axis perpendicular to the plane of the meridian is $r\frac{dp}{rd\psi} = \frac{dp}{d\psi}$.

The polar equations of hydro-dynamics are therefore

$$(X\cos\phi + Y\sin\phi)\cos\psi + Z\sin\psi + \frac{d\psi^2}{dt^2} + r\cos^2\psi\frac{d\phi^2}{dt^2} - \frac{d^2r}{dt^2} = \frac{dp}{dr},$$

$$(Y\cos\phi - X\sin\phi)\,r\cos\psi - \frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\} = \frac{dp}{d\phi},$$

$$Zr\cos\psi - (X\cos\phi + Y\sin\phi)\,r\sin\psi - r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2} - \frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\} = \frac{dp}{d\psi}.$$

These are substantially the same as those given by Mr. AIRY in his article on *Tides and Waves*, in the *Encyclopedia Metropolitana*, p. 264."

V. *Solution by the* PROPOSER.

The complete differential equations of the motion of the sea or atmosphere, referred to polar coordinates, are regarded, justly, as one of the most brilliant results that we owe to the genius of Laplace; and yet they are found to be a "stumbling-block" in the way of young mathematicians, from the hideously repulsive form in which they are deduced, by transformation, from fixed rectilinear coordinates, by Laplace himself, and by his followers. Any attempt, therefore, to write down these equations at sight, from elementary geometrical and mechanical principles, will be regarded as useful. According to the self-evident principle of D'Alembert, all problems of Dynamics are reducible to problems of Statics, by introducing velocities and accelerating forces, equal and opposite to the existing velocities and accelerating forces. Now, the most general equations of equilibrium, of any system, are the following, six in number:—

$$X = 0,\quad Y = 0,\quad Z = 0,\quad L = 0,\quad M = 0,\quad N = 0 \;\ldots\ldots\ldots (1),$$

where X, Y, Z are the sums of the external forces resolved along three rectangular axes; and L, M, N are the sums of the couples (or twists) of those forces round the axes of X, Y, Z respectively.

The corresponding dynamical equations are these:—

$$X - \frac{d^2x}{dt^2} = 0,\quad Y - \frac{d^2y}{dt^2} = 0,\quad Z - \frac{d^2z}{dt^2} = 0,\quad L - \frac{d}{dt}\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right) = 0,$$

$$M - \frac{d}{dt}\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right) = 0,\quad N - \frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = 0 \ldots\ldots (2).$$

In the case of an incompressible fluid these equations become

$$\frac{dp}{dx} = X - \frac{d^2x}{dt^2},\quad \frac{dp}{dy} = Y - \frac{d^2y}{dt^2},\quad \frac{dp}{dz} = Z - \frac{d^2z}{dt^2} \;\ldots\ldots\ldots (3),$$

where p is the common pressure of the fluid (equal in all directions) at any point (x, y, z). It will be noted, that the last three equations of (2), depending on couples, disappear; because, in consequence of the mobility of the particles of the fluid *inter se*, internal couples, or twists, become impossible.

The Laplacian equations of motion, in polar coordinates, are usually deduced from (3), by transformation of the coordinates, from x, y, z, re-

ferred to fixed axes, where the axis of x is the axis of rotation; that of y an axis perpendicular to x, and fixed in space; and that of z, an axis perpendicular to those of x, y; to r, θ', ϕ', where r is the radius vector, θ' is the north polar distance, and ϕ' is the angular distance from the plane of x, y, of the meridian of any moving particle.

Instead of referring the forces to fixed coordinates, I refer them to the following movable rectangular coordinates:—

Axis of x'.—Let R denote the sum of the forces at any point, acting along the radius vector (*negative* towards the centre, and *positive* from it).

Axis of y'.—Let S denote the sum of the forces at any point acting in the meridional moving plane, and perpendicular to r (*positive* towards the equator, and *negative* towards the pole).

Axis of z'.—Let T denote the sum of the forces at any point acting perpendicularly to the two former directions, or in the direction of the tangent to the small circle of latitude (*negative* against the rotation, and *positive* with it. Let r, θ', ϕ' denote the polar coordinates in their most general form. The alteration in pressure produced by a change in r is similar to that produced by a change in x, y, z of the first three of equations (2) (because they are all linear magnitudes), and denotes a *force* acting to or from the centre; but the alteration in pressure produced by a change in angular direction by a change in θ' or ϕ' is no longer a *force*, but a *couple*, tending to turn the fluid round the centre. Thus,

$\frac{dp}{dr}$ is a *force* acting in the direction of the radius vector;

$\frac{dp}{d\theta'}$ is a *couple*, acting always in the moving meridional plane, and whose axis moves perpendicular to that plane;

$\frac{dp}{d\phi}$ is a *couple*, acting always round the axis of rotation, and parallel to the equatorial plane.

It is evident that, if D'Alembert's equations (2) are satisfied—

(1) for *forces* acting along the radius vector;

(2) for *couples* acting in the meridional plane in every possible position of that plane;

(3) for couples acting always round the axis of rotation;

then complete dynamical equilibrium will be secured.

We may discount all the mechanical consequences of the rotation by introducing the centrifugal force, leaving only the geometrical consequences of the rotation, in the problem. The geometrical effect of the rotation is expressed by writing $\phi' = nt + \phi'$,

where n is the angular velocity of the earth's rotation.

The components of the velocity of any particle along R, S, T are

$$\frac{dr}{dt}, \quad r\frac{d\theta'}{dt}, \quad r\sin\theta'\left(n + \frac{d\phi'}{dt}\right).$$

The centrifugal force affects the directions R, S only, and does not enter into T. The centrifugal force in the direction of R is, obviously,

$$\frac{1}{r}\left\{\frac{r^2 d\theta'^2}{dt^2} + r^2\sin^2\theta'\left(n + \frac{d\phi'}{dt}\right)^2\right\}.$$

From this, and from the first three equations (2) we find, at sight,

$$\frac{dp}{dr} = \mathrm{R} - \frac{d^2r}{dt^2} + \frac{rd\theta'^2}{dt^2} - r\sin^2\theta'\left(n + \frac{d\phi'}{dt}\right)^2 \ldots\ldots\ldots\ldots (\mathrm{A}).$$

The centrifugal force in the direction of S is, obviously,

$$r \sin\theta' \cos\theta' \left(n + \frac{d\phi'}{dt}\right)^2.$$

The sixth of equations (2) therefore becomes, remembering that

$$y' \frac{dx'}{dt} - x' \frac{dy'}{dt} = r^2 \frac{d\theta'}{dt},$$

and, equating couples in the meridional plane,

$$\frac{dp}{d\theta'} = Sr - \frac{d}{dt}\left(r^2 \frac{d\theta'}{dt}\right) + r^2 \sin\theta' \cos\theta' \left(n + \frac{d\phi'}{dt}\right)^2 \;\ldots\ldots\; \text{(B)}.$$

If we now equate the couples in the equatorial plane, we find, since

$$z\frac{dy}{dt} - y\frac{dz}{dt} = r^2 \sin^2\theta' \left(n + \frac{d\phi'}{dt}\right),$$

$$\frac{dp}{d\phi'} = Tr \sin\theta' - \frac{d}{dt}\left\{ r^2 \sin^2\theta' \left(n + \frac{d\phi'}{dt}\right) \right\} \;\ldots\ldots\ldots\ldots\; \text{(C)}.$$

The three equations just found from elementary principles are *exact equivalents* of the Laplacian differential equations, which are expressed by AIRY (in his *Tides and Waves*, p. 264) in the form given in the Question.

1401. (By the EDITOR).—Eliminate x from the equations

$$x^5 + px^2 + q + xr = 0, \quad x^4 + dx^3 + cx^2 + bx + a = y,$$

and exhibit the result in the form

$$y^5 + Ay^4 + By^3 + Cy^2 + Dy + E = 0,$$

giving the values of A, B, C, D, E in terms of p, q, r, a, b, c.

Solution by the late Professor CLIFFORD, F.R.S.

If we multiply the second equation by x, and then each equation in succession by $x^4, x^3, \ldots, x^0$, we shall have *ten* simple equations from which to eliminate the *ten* quantities $x^9, x^8, x^7, \ldots, x^1, x^0$; and the result is (if we put a for $a-y$) the determinant

$$\begin{vmatrix} 1 & d & c & b & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & d & c & b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d & c & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & d & c & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & d & c & b & a & 0 \\ 1 & 0 & 0 & p & q & r & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & p & q & r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & p & q & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & p & q & r & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & p & q & r \end{vmatrix} = 0.$$

This, however, I have not had the courage even to attempt to work out. It will evidently be divisible by r.

NOTE ON THE SOLUTION OF QUESTION 1401, *by the* EDITOR.

The foregoing solution is one of many early unpublished contributions sent to us by a mathematician whose recent loss to science our readers, in common with the whole scientific world, have especial reason to deplore. In now publishing, what the small space at our disposal when the solution was sent, *sixteen* years ago, did not then permit us to publish, we would join in a hope expressed by Mr. CLIFFORD in an accompanying letter, that some contributor familiar with the manipulation of determinants would develope the result in the form required by the question. The developed form may then be compared with the result obtained by Mr. HARLEY in his solution of the question, as given on pp. 38—40 of Vol. I. of our *Mathematical Reprints*. The question is connected with the problem of the reduction of the general quintic to a trinomial form, and has an especial reference to the transformation effected, so early as 1786, by ERLAND S. BRING, a Swedish mathematician. Reference may also be made to pages 8 and 67 of Vol. I. of the *Reprints*.

We may remark that the letters sent by Mr. CLIFFORD with his solutions were even more interesting and valuable than the solutions themselves, and that the very earliest of them, sent at a time when, in speaking of a kite-string problem, he remarks, in passing, that he had not yet left off flying kites, showed unmistakeable signs of the originality and power by which he was afterwards so pre-eminently distinguished. In a letter bearing date "Exeter, Sep. 10, 1863," Mr. CLIFFORD returns again to Question 1401, and states that, though he has several times tried to complete his solution, "the determinant has baffled me, from my not being accustomed to the manipulation of determinants; but I hope soon to be settled at Cambridge, and then I will try again."

Since the publication of Mr. CLIFFORD's Solution, with editorial note thereon, in the *Educational Times* for June 1879, several incomplete attempts at a development of the determinant have been sent in. Of these the only one that we can spare room for is contained in the following note by Mr. WOOLHOUSE:—

"I would remark that, in the foregoing Solution to Question 1401, the determinant stated is unnecessarily extended. The tenth simple equation being the only one that involves the tenth quantity x^0, both it and that quantity should evidently be suppressed, and the determinant, as indicating the elimination of nine quantities $x^9, x^8, \dots x^1$, from nine simple equations, stated thus:—

$$\begin{vmatrix} 1 & d & c & b & a & 0 & 0 & 0 & 0 \\ 0 & 1 & d & c & b & a & 0 & 0 & 0 \\ 0 & 0 & 1 & d & c & b & a & 0 & 0 \\ 0 & 0 & 0 & 1 & d & c & b & a & 0 \\ 0 & 0 & 0 & 0 & 1 & d & c & b & a \\ 1 & 0 & 0 & p & q & r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & p & q & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & p & q & r & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & p & q & r \end{vmatrix} = 0.$$

The development of this determinant may be obtained by eliminating directly from the nine equations. The five quantities x^9, x^8, x^7, x^6, x^1 admit of very easy elimination, thereby reducing the work down to four equations amongst four quantities, that is, a determinant of the fourth order, the development of which consists of twenty-four products, written out according to the well known-formulæ."

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Scott, J., Judge of Ohio Supreme Court.
Scott, R. F., M.A.; Fell. St. John's Coll., Camb.
Seitz, Prof., M.A.; Kirksville, Mo., U.S.
Selby, A. L., Leigh, Manchester.
Serret, Professor; Paris.
Sharp, W. J. C., M.A.; Grosvenor Square.
Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham.
Shepherd, A. J. P.; Queen's College, Oxford.
Sides, J. J.; Rue des Vieillards, Boulogne.
Siverly, **Walter**; Oil City, Pennsylvania.
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Vincenzo, **Jacobini**; University of Rome.
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Wolstenholme, Rev. J., M.A.; Professor of Mathematics in Cooper's Hill College.
Woolhouse, W. S. B., F.R.A.S., &c.; London.
Wright, Dr. S. H., M.A.; Penn Yan, New York.
Wright, Rev. W. J., Ph.D.; Pennsylvania.
Young, **John**, B.A.; Academy, Londonderry.

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Mathematical Papers, &c.

Solved Questions.

CORRIGENDA IN VOL. XXXII.

Page 26, line 2, *for* $\frac{R}{4r}$ *read* $\frac{r}{4R}$.

Page 38, lines 3 to 6 from bottom, *for* Δ *read* triangles.

Page 70, Fig. to Quest. 5799, the letter E should be at vertex of arc DF.

Page 70, line 18, *read* $S = ab(\theta-\sin\theta)$.

Page 70, line 5 from bottom, interchange the indices $\frac{3}{2}$ and $\frac{1}{2}$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

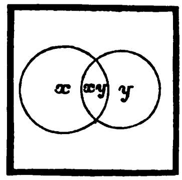
5626. (By the late Professor CLIFFORD, F.R.S.)—The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle; find the envelope of all the binormal circles of one system.

Solution by J. J. WALKER, M.A.

There being four points with respect to which as centres of inversion a bicircular quartic is its own inverse, there will be four systems of doubly normal circles; viz., the circle having as its centre the intersection (T) of tangents at two inverse points relatively to one centre, and passing through those points, will be a circle of one system. Dr. CASEY, in his well-known memoir on these curves, regarded as envelopes of a "generating circle" (K), the centre of which lies on a conic (F) while it cuts orthogonally a fixed circle J, has given a construction for the point T as the intersection of a tangent to F at the centre of K (touching the quartic at a pair of inverse points), with its polar relatively to J. Dr. CASEY has also shown, if F and J be $ax^2+by^2+cz^2=0$, $a'x^2+b'y^2+c'z^2=0$, that the locus of T (ξ, η, ζ) will be $$\frac{a''}{\xi^2}+\frac{b''}{\eta^2}+\frac{c''}{\zeta^2}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$ where $a''=\dfrac{(bc'-b'c)^2}{b'c'}$, $b''=\ldots$, $c''=\ldots$ Now (Quest. 6050) the circle with centre T and orthogonal to K will also be orthogonal to J; its equation will therefore be $$\xi S+\eta S'+\zeta S''=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$ where $S=-a\sin A\,x^2+b\sin A\,y^2+c\sin A\,z^2-2a\sin C\,zx-2a\sin B\,xy$, $S=\ldots$, $S''=\ldots$ The envelope of (2), when ξ, η, ζ satisfy (1), may be found by forming the condition that two conics should touch; and will be $$27a''b''c''\,S^2S'^2S''^2+(a''S^2+b''S'^2+c''S''^2)^3=0,$$ a curve of the twelfth order.

ON A QUESTION IN PROBABILITIES. *By* A. MACFARLANE, D.Sc., F.R.S.E.

In a work which I have just published, entitled *Principles of the Algebra of Logic*, I endeavour to show that there is an Algebra of Quality, which is a generalized form of the Algebra of Quantity, and that this Algebra supplies numerous methods of singular beauty and power for investigating problems which relate to Necessity or to Probability. The following theorem appears to me to throw light upon the question in Probability, TODHUNTER'S *Algebra*, which was brought forward by Mr. ARTEMAS MARTIN, and discussed by eminent mathematicians in Vol. XVII. of the *Reprint* (pp. 77, 89, 90, 91). It consists in finding the arithmetical value of $\frac{xy}{x}$. The meaning of this expression is shown by the diagram.



The collection of individuals forming the subject of discourse is represented by the part of the page within the square: those which have a character x by the part inside the one circular line, and those which have the character y by the part inside the other circular line; those having both x and y by the part inside both lines; those having neither by the part outside.

Now $xy = xy$, therefore $\quad y = \frac{xy}{x}$.

Expand $\frac{xy}{x}$ in terms of the parts of the universe (represented by the square) formed by means of x and y.

$$\frac{xy}{x} = axy + bx(1-y) + c(1-x)y + d(1-x)(1-y).$$

Let $x = 1$ and $y = 1$, then $\quad a = 1$.
Let $x = 1$ and $y = 0$, then $\quad b = 0$.
Let $x = 0$ and $y = 1$, then $\quad c = \frac{0}{0}$.
Let $x = 0$ and $y = 0$, then $\quad d = \frac{0}{0}$.

Therefore $$\frac{xy}{x} = xy + \tfrac{0}{0}(1-x);$$

hence $\quad y = xy + \frac{0}{0}(1-x)$, and $\bar{y} = \overline{xy} + \frac{0}{0}(\bar{1}-\bar{x})$,

where the bar denotes that the arithmetical value of the quality is taken.

The problem referred to is as follows:—

"A. says that B. says that a certain event took place; required the probability that the event did take place, p_1 and p_2 being A.'s and B.'s respective probabilities of speaking the truth."

The different solutions are the following:—

(1) From TODHUNTER'S *Algebra*, $p_1p_2 + (1-p_1)(1-p_2)$.

(2) Mr. ARTEMAS MARTIN'S, $p_1\{p_1p_2 + (1-p_1)(1-p_2)\}$.

(3) Mr. WOOLHOUSE'S and American mathematicians', p_1p_2.

(4) Professor CAYLEY'S, $p_1p_2 + \beta(1-p_1)(1-p_2) + \kappa(1-\beta)(1-p_1)$,

where β is the chance, on the supposition of the incorrectness of A.'s statement, that B. told A. that the event did *not* happen, and $1-\beta$ that he did not tell him at all, and κ is the antecedent probability.

Let U = statements of A. about B.'s statements about an event taking place, x = which truly reported a statement by B., y = which truly reported the event.

Then $x = \overline{p}_1$, and $xy = \overline{p}_1\overline{p}_2$; therefore, by means of the theorem proved, $\overline{y} = \overline{p}_1\overline{p}_2 + \frac{0}{0}(\overline{1-p_1})$, therefore $y > \overline{p}_1\overline{p}_2$, and $< \overline{p}_1\overline{p}_2 + \overline{1} - \overline{p}_1$.

Mr. TODHUNTER assumed that $\frac{0}{0} = 1 - p_2$; and Mr. WOOLHOUSE that $\frac{0}{0} = 0$.

The above solution is contradictory to the first three, and agrees with the fourth, without introducing more than one unknown quantity. But, by means of the theorem referred to, we can find the solution when there are n persons involved in the tradition—

A_1 says that A_2 says that A_3 says . . . that A_n says a certain event took place. The probabilities of $A_1, A_2, \ldots A_n$ speaking the truth are $p_1, p_2, \ldots p_n$ respectively. Required the probability that the event took place. Now let

U = series of statements of A_1 about A_2 saying &c.,
x_1 = which truly reported a statement of A_2,
... ...
x_n = " " " the event.

Now, $$x_n = \frac{x_1x_2 \ldots x_n}{x_1x_2 \ldots x_{n-1}} = x_1x_2 \ldots x_n + \tfrac{0}{0}(1 - x_1x_2 \ldots x_{n-1}) \text{ by the theorem,}$$
$$= \overline{p}_1\overline{p}_2 \ldots \overline{p}_n + \tfrac{0}{0}(\overline{1} - \overline{p}_1\overline{p}_2 \ldots \overline{p_{n-1}}) \text{ by the data.}$$

Cor. 1.—Suppose that each always reports truly. Then
$$x_n = 1 + \tfrac{0}{0} \times 0 = 1.$$

Cor. 2.—Suppose that each always reports truly excepting A_n. Then
$$x_n = \overline{p}_n.$$

Cor. 3.—Suppose that A_n always speaks falsely, then
$$x_n = \tfrac{0}{0}(\overline{1} - \overline{p}_1\overline{p}_2 \ldots \overline{p_{n-1}}).$$

Cor. 4.—Suppose that any other than A_n always speaks falsely, then $x_n = \frac{0}{0}$; that is, the probability is quite indefinite.

Cor. 5.—Suppose that each $\overline{p} = \frac{1}{2}$, then
$$x_n = (\tfrac{1}{2})^n + \tfrac{0}{0}[1 - (\tfrac{1}{2})^{n-1}];$$
which, when n is infinite, is equal to $\frac{0}{0}$, that is, is perfectly indefinite.

5181. (By Professor TOWNSEND, F.R.S.)—The particulars of the motion of a system of material particles, connected by any relations and acted on by any forces which are functions of configuration only, being supposed such that the system would ultimately reach with exhausted energy of motion a position of unstable equilibrium against the action of the forces; show, generally, that the position could ultimately be attained to only at the expiration of an infinite time.

Solution by the PROPOSER.

It is a well known general property of the motion of every system of particles, circumstanced as above supposed, that if, in any position of the

system, the velocities of the several particles were all changed into their opposites, the system would return back into every position through which it had previously passed, in the same time it had taken to pass from the previous to the subsequent position; and as the system would evidently take an infinite time to pass without initial energy of motion from any position of equilibrium to any other whatsoever, so, therefore, would it take an infinite time to reach without terminal energy of motion any position of equilibrium from any other whatsoever; and therefore, &c.

A more formal and direct, but not more valid, proof of the property may be given as follows:—

The system being supposed, near the termination of its motion, to approach, with nearly exhausted energy of motion, so near to the supposed position of unstable equilibrium under the action of the forces, that all powers above the first of the relative coordinates and of the absolute velocities of the several particles may be neglected in the equations of their motion; if, then α, β, γ, &c., be the values corresponding to the position of equilibrium of the several independent coordinates of position of the system, and $\alpha+u$, $\beta+v$, $\gamma+w$, &c. their values corresponding to the near position, the differential equations for the determination of u, v, w, &c., as functions of the time, in number n equal to that of the independent coordinates, will, as is well known, be of the form

$$a_1\frac{d^2u}{dt^2}+b_1\frac{d^2v}{dt^2}+c_1\frac{d^2w}{dt^2}+\&\text{c.}\ +e_1u+f_1v+g_1w+\&\text{c.}=0,$$

$$a_2\frac{d^2u}{dt^2}+b_2\frac{d^2v}{dt^2}+c_2\frac{d^2w}{dt^2}+\&\text{c.}\ +e_2u+f_2v+g_2w+\&\text{c.}=0,$$

$$a_3\frac{d^2u}{dt^2}+b_3\frac{d^2v}{dt^2}+c_3\frac{d^2w}{dt^2}+\&\text{c.}\ +e_3u+f_3v+g_3w+\&\text{c.}=0,$$

&c., where $a_2=b_1$, &c., and $e_2=f_1$, &c., and their complete integrals of the form

$$u=\Sigma(\mathrm{P}\,.\,\epsilon^{+\lambda t})+\Sigma\,(p\,.\,\epsilon^{-\lambda t}),$$
$$v=\Sigma(\mathrm{Q}\,.\,\epsilon^{+\lambda t})+\Sigma\,(q\,.\,\epsilon^{-\lambda t}),$$
$$w=\Sigma(\mathrm{R}\,.\,\epsilon^{+\lambda t})+\Sigma\,(r\,.\,\epsilon^{-\lambda t}),\ \&\text{c.},$$

where the several coefficients P, Q, R, &c. involve n, and the several p, q, r, &c. the remaining n, of the $2n$ arbitrary constants of integration, and where the several values of λ^2 are the several roots, necessarily all real, of the symmetrical determinantal equation,

$$\begin{vmatrix} a_1\lambda^2+e_1 & b_1\lambda^2+f_1 & c_1\lambda^2+g_1 & \&\text{c.} \\ a_2\lambda^2+e_2 & b_2\lambda^2+f_2 & c_2\lambda^2+g_2 & \&\text{c.} \\ a_3\lambda^2+e_3 & b_3\lambda^2+f_3 & c_3\lambda^2+g_3 & \&\text{c.} \\ \&\text{c.} & \&\text{c.} & \&\text{c.} & \&\text{c.} \end{vmatrix}=0,$$

all of which will, as is well known, be positive for every position of perfectly unstable, and negative for every position of perfectly stable, equilibrium of the system.

Now, since by hypothesis u, v, w, &c., must all $=0$, for all values of t from the time τ of reaching the supposed position of unstable equilibrium up to infinity, all the terms involving the several positive values of λ must be absent from the above values of u, v, w, &c., which otherwise could not

continue small, much less vanish, for large values of t, and we shall have accordingly

$$u = p_1 \cdot \epsilon^{-\lambda_1 t} + p_2 \cdot \epsilon^{-\lambda_2 t} + p_3 \cdot \epsilon^{-\lambda_3 t} + \text{\&c.},$$
$$v = q_1 \cdot \epsilon^{-\lambda_1 t} + q_2 \cdot \epsilon^{-\lambda_2 t} + q_3 \cdot \epsilon^{-\lambda_3 t} + \text{\&c.},$$
$$w = r_1 \cdot \epsilon^{-\lambda_1 t} + r_2 \cdot \epsilon^{-\lambda_2 t} + r_3 \cdot \epsilon^{-\lambda_3 t} + \text{\&c. \&c.},$$

which, even when the determinant $[p_1, q_2, r_3, \text{\&c.}] = 0$, could not in general be all $= 0$, for any common value τ of t, much less for all values of t between τ and ∞, except when $\tau = \infty$; and therefore, &c.

5981. (By Artemas Martin, M.A.)—Prove that, if

$$S_m = x^m + y^m + z^m, \quad P_m = x^m y^m + x^m z^m + y^m z^m, \text{ and } Q = xyz,$$

then $S_n = S_1 S_{n-1} - P_1 S_{n-2} + QS_{n-3}$, $P_n = P_1 P_{n-1} - QS_1 P_{n-2} + Q^2 S_{n-3}$.

Solution by H. L. Orchard, B.A., L.C.P.; J. Hammond, M.A.; *and others.*

We have $x^n = S_1 x^{n-1} - P_1 x^{n-2} + Q x^{n-3}$, and similarly for y and z; therefore, by addition, $S_n = S_1 S_{n-1} - P_1 S_{n-2} + QS_{n-3}$. In like manner,

$$\frac{1}{x^n} = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\frac{1}{x^{n-1}} - \left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right)\frac{1}{x^{n-2}} + \frac{1}{xyz} \cdot \frac{1}{x^{n-3}},$$

or

$$\frac{1}{x^n} = \frac{P_1}{Q} \cdot \frac{1}{x^{n-1}} - \frac{S_1}{Q} \cdot \frac{1}{x^{n-2}} + \frac{1}{Q} \cdot \frac{1}{x^{n-3}}.$$

Adding like expressions involving y and z, and multiplying by Q^n, we obtain

$$P_n = P_1 P_{n-1} - S_1 Q P_{n-2} + Q^2 P_{n-3}.$$

5970. (By E. B. Elliott, M.A.)—If S_r denote the infinite series

$$1 \cdot 2 \cdot 3 \ldots r + \frac{2 \cdot 3 \cdot 4 \ldots (r+1)}{2!} + \frac{3 \cdot 4 \cdot 5 \ldots (r+2)}{3!} + \ldots\ldots;$$

prove that

$$S_r = (2r-1) S_{r-1} - (r-1)(r-2) S_{r-2}.$$

Solution by T. R. Terry, M.A.; R. Knowles, L.C.P.; *and others.*

Let $S_r = u_1 + u_2 + \ldots$ and $\phi(S_r) = S_r - (2r-1) S_{r-1} + (r-1)(r-2) S_{r-2}$; then

$$\phi(u_n) = \frac{(n+r-3)!}{n!\,(n-1)!}[n^2 - n - (n-r-2)] = \frac{(n+r-3)!}{(n-1)!\,(n-2)!} - \frac{(n+r-2)!}{n!\,(n-1)!}.$$

So $\quad \phi(u_1) = r! - (2r-1)(r-1)! + (r-1)(r-2)(r-2)! = -(r-1)!$

$$\phi(u_2) = (r-1)! - \frac{(r-2)!}{2}, \quad \phi(u_3) = \frac{(r-2)!}{2!} - \frac{(r-3)!}{3!\,2!},$$

therefore, adding, $\qquad \phi(S) = 0.$

5967. (By Professor DE LONCHAMPS.)—Etant donnés deux points P et P′ à l'intérieur d'une circonférence, sur un même diamètre et à égale distance du centre; on propose de mener deux parallèles PQ, P′Q′ terminées à la circonférence, et telles que le trapèze PQP′Q′ soit maximum.

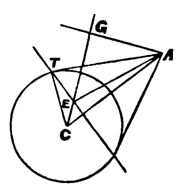
Solution by G. TURRIFF, M.A.; W. J. MACDONALD, M.A.; *and others.*

Le trapèze PQP′Q′, qui sera un parallelogramme, est le double de chacun des triangles PQP′, PQ′P′, dont l'aire variera avec la hauteur du triangle et par conséquent le trapèze sera maximum quand QQ′ est le diamètre perpendiculaire au diamètre donné.

1373. (By the late T. T. WILKINSON, F.R.A.S.)—Given a circle (C) and *any* point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that, if LEM be drawn cutting the circle in L, M, we may always have

$$AE^2 = LE \cdot EM \pm BA \cdot AD.$$

Solution by the late Professor CLIFFORD, F.R.S.



1. For the upper sign, when A is *without* the circle, and the lower, when A is *within*, E lies on the polar of A.

For $AE^2 = AG^2 + GE^2 = AC^2 - CG^2 + GE^2$
$= AC^2 - CG^2 + (CG^2 + CE^2 - 2\,CE \cdot CG)$
$= AC^2 + CE^2 - 2\,CT^2$
$= BA \cdot AD - LE \cdot EM,$

when E is within the circle, as in the figure; when E is without the circle, $AE^2 = LE \cdot EM + BA \cdot AD$. By interchanging A and E, we get the second case.

2. For the same; E may obviously lie on a circle with CA as diameter.

3. For the upper sign, when A is *within* the circle, and the lower, when A is *without*; bisect CA in P, and with P as centre describe a circle whose radius is $\sqrt{(r^2 - 3CP^2)}$; E may lie on this circle. For, if EQ be perpendicular to AC, then $r^2 - 3CP^2 = PE^2 = PQ^2 + QE^2$,

or $$2r^2 - \tfrac{3}{2}AC^2 = CE^2 + AE^2 - 2AP^2,$$

therefore $AE^2 = (r^2 - CE^2) \pm (r^2 \mp CA^2) = LE \cdot EM + BA \cdot AD.$

4. In the same cases, E may evidently lie on a straight line through A perpendicular to CA.

[This solution was received in 1863. A solution by Professor CAYLEY of the same Question may be seen on p. 6 of Vol. I. of our *Reprints*.]

5750. (By H. Fortey, M.A.)—A large number of equal spheres being placed in the closest possible contact, find the ratio of their aggregate volume to the space they take up.

Solution by G. Heppel ; Prof. Vose, M.A. ; *and others.*

It will be found that a sphere can be touched by 12 others of the same size *and not more ;* and therefore, when a number of spheres are in the closest possible contact, *each* will be touched by 12 others ; and, the tangent planes being drawn at the contacts, each sphere will be circumscribed by a polyhedron, which is the space peculiar to it, or which it may be said to take up. The question will therefore be solved when the ratio of a sphere to its circumscribing polyhedron has been determined. Now a simple application of geometry of three dimensions (the work is not given here on account of the difficulty of drawing the illustrative diagram on a plane surface) shows that the polyhedron is the rhombic dodecahedron, and that, if a be the radius of the sphere, a side of a rhombus is $\sqrt{\frac{3}{2}}\,a$, and the acute angle between two sides $\cos^{-1}\frac{1}{3}$. Therefore the area of a rhombus is $\sqrt{2}\,a^2$, and the volume of the dodecahedron is equal to that of the 12 pyramids having the rhombuses for their bases, and the radius of the sphere for their common height $= 12\sqrt{2}\,a^2 \times \frac{1}{3}a = 4\sqrt{2}\,a^3$.

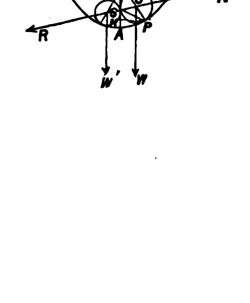
Therefore Vol. of sphere : Vol. of dodecahedron

$$= \tfrac{4}{3}\pi a^3 : 4\sqrt{2}\,a^3 = \tfrac{1}{6}\pi\sqrt{2} : 1 = \cdot 740480 : 1.$$

Thus the spheres actually fill very nearly three-fourths of the space they take up.

5460. (By Professor Wolstenholme, LL.D.)—Two spheres of densities ρ, σ, and whose radii are a, b, rest in a paraboloid of revolution, whose axis is vertical, and touch each other at the focus; prove that $\rho^3 a^{10} = \sigma^3 b^{10}$.

Solution by R. Tucker, M.A. ; R. E. Riley, B.A. ; *and others.*



Consider the equilibrium of the sphere (O). It is kept at rest by the mutual action R, by the reaction of the curve along the normal (which cuts the axis in T), and by its weight, therefore $\dfrac{\frac{4}{3}g\rho\pi a^3}{\text{R}} = \dfrac{\sin\theta}{\sin\phi}$ ....(1),

if $\angle\,\text{TOS} = \theta$, $\angle\,\text{OTS} = \phi$, by property of parabola (because ST = SP), $\theta = 2\phi$. Let co-ordinates of P be x', y', then

$$\cos 2\phi = \frac{\mu - x'}{\mu + x'}$$

(if equation to the curve is $y^2 = 4\mu x$), and

$$\cos^2\phi = \frac{\mu}{\mu + x'} \text{ ....(2) ;}$$

also $\quad (x' + \mu) \sin 2\phi = y' = a(\sin\phi + \sin 3\phi),$

therefore $\quad x' + \mu = 2a\cos\phi;$

hence, by (2), $\quad 2a\cos\phi = \mu\sec^2\phi, \quad$ and $\cos^3\phi = \dfrac{\mu}{2a}.$

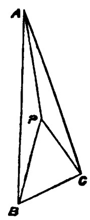
Now $$\frac{\sin\theta}{\sin\phi} = 2\cos\phi = 2\left(\frac{\mu}{2a}\right)^{\frac{1}{3}}.$$

Hence, from (1), $\dfrac{\rho a}{\sigma b^3} = \left(\dfrac{b}{a}\right)^{\frac{1}{3}}$, therefore $\rho^3 a^{10} = \sigma^3 b^{10}$.

5933. (By the Editor.) — A right-angled triangle, placed with its hypotenuse vertical, is circumscribed by a rectangular hyperbola having its centre at the middle of the lower side: prove that a body's times of falling from any point on the curve to the ends of the lower side are equal.

Solution by D. Edwardes.

Let ABC be the triangle, and P any point on the curve. Then, since the angles that any chord of a rectangular hyperbola subtends at the ends of a diameter are either equal or supplementary, we have $\angle$PBA = PCA = θ suppose.



Now the times down PB, PC respectively are

$$t_1 = \sqrt{\frac{2}{g}\cdot\frac{\text{PB}}{\cos\theta}}, \quad t_2 = \sqrt{\frac{2}{g}\cdot\frac{\text{PC}}{\cos(\text{A}+\theta)}}.$$

But, since BCA is a right angle, we have

$$\frac{\text{PB}}{\cos\theta} = \frac{\text{PC}}{\sin(\text{B}-\theta)} = \frac{\text{PC}}{\cos(\text{A}+\theta)};$$

hence $t_1 = t_2$; that is, the time down PB is equal to the time down PC.

If the angles PCA, PBA are supplementary, the times vary as

$$\frac{\text{PB}}{\cos(\text{ABP})} \text{ and } \frac{\text{PC}}{\cos(\text{ACP}-\text{A})}.$$

But $$\frac{\text{PB}}{\sin(2\pi-\frac{1}{2}\pi-\text{ACP})} = \frac{\text{PC}}{\sin(\text{ABP}-\text{B})},$$

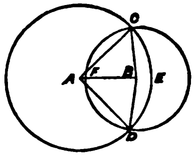
whence $$\frac{\text{PB}}{\cos(\text{ABP})} = \frac{\text{PC}}{\cos(\text{ACP}-\text{A})}.$$

5795. (By Professor Crofton, F.R.S.)—A circle is thrown at random on a given fixed circle traced on the ground. Find the mean value (1) of the arc; (2) of the area covered by it. [The cases where the circles do not meet are omitted.]

I. *Solution by* E. B. SEITZ.

Let A be the centre of the given fixed circle, B the centre of the other circle, and C, D the intersections of the circles.

1. Let AC = R, BC = r, AB = x, $\angle$BAC = θ, $\angle$ABC = ϕ, and $\angle$ACB = ψ. Then we have arc CED = 2θR,



$$\cos\theta = \frac{R^2+x^2-r^2}{2Rx},\quad \cos\phi = \frac{r^2+x^2-R^2}{2rx} \ldots (1,\ 2),$$

$$\cos\psi = \frac{R^2+r^2-x^2}{2Rr} \ldots\ldots\ldots\ldots\ldots (3),$$

$$x\sin\theta = r\sin\psi,\quad x\cos\phi = r - R\cos\psi \quad \ldots\ldots\ldots\ldots (4,\ 5).$$

An element of surface at the distance x from A is $2\pi x dx$; moreover, in order that the circles may meet, the limits of x must be $R-r$ and $R+r$. Hence the mean value of the arc CED is

$$z_1 = \int_{R-r}^{R+r} 2\theta R\,.\,2\pi x\,dx \div \int_{R-r}^{R+r} 2\pi x\,dx = \frac{1}{2r}\int_{R-r}^{R+r} 2\theta x\,dx.$$

But
$$\int 2\theta x\,dx = \theta x^2 - \int x^2 d\theta.$$

From (1) and (2), $\quad Rx\sin\theta\,d\theta = -r\cos\phi\,dx \ldots\ldots\ldots\ldots\ldots\ldots (6).$

From (4) and (6), $\quad R\sin\psi\,d\theta = -\cos\phi\,dx \ldots\ldots\ldots\ldots\ldots\ldots (7),$

and, from (3), $\quad Rr\sin\psi\,d\psi = x\,dx \ldots\ldots\ldots\ldots\ldots\ldots (8).$

From (7) and (8), $\quad x\,d\theta = -r\cos\phi\,d\psi \ldots\ldots\ldots\ldots\ldots\ldots (9),$

and, from (5) and (9), $\quad x^2 d\theta = -r\,(r - R\cos\psi)\,d\psi;$

therefore
$$\int 2\theta\,x\,dx = \theta x^2 + \psi r^2 - Rr\sin\psi.$$

Hence, observing that, when $x = R-r$, $\theta = 0$ and $\psi = 0$, and when $x = R+r$, $\theta = 0$ and $\psi = \pi$, we find $z_1 = \frac{1}{2}\pi r = \frac{r}{R}$ of the quadrant of the circle A. When $r > R$, the limits of x are $r-R$ and $r+R$; those of θ, π and 0; and those of ψ, 0 and π; and we find the mean value to be

$$z_2 = \pi R\left(1 - \frac{R}{2r}\right).$$

2. The area covered by the circle B is area CEDF = $\theta R^2 + \phi r^2 - Rr\sin\psi$, and its mean value is

$$u_1 = \int_{R-r}^{R+r} (\theta R^2 + \phi r - Rr\sin\psi)\,2\pi x\,dx \div \int_{R-r}^{R+r} 2\pi x\,dx$$

$$= \frac{1}{4Rr}\int_{R-r}^{R+r} (\theta R^2 + \phi r^2 - Rr\sin\psi)\,2x\,dx.$$

In a manner similar to that above, it may be shown that

$$\int 2\phi\,x\,dx = \phi x^2 + \psi R^2 - Rr\sin\psi,$$

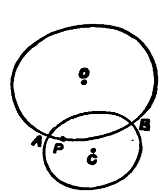
and, from (8), $\int 2x \sin\psi\, dx = \int 2Rr \sin^2\psi\, d\psi = \psi Rr - \frac{1}{2} Rr \sin 2\psi.$

Hence, since the limits of ϕ are π and 0, we find $u_1 = \pi r^2 \left(\frac{1}{2} - \frac{R}{4r}\right)$.

When $r > R$, the limits of ϕ are 0 and 0, and we find the mean value to be

$$u_2 = \pi R^2 \left(\frac{1}{2} - \frac{R}{4r}\right).$$

II. *Solution by the* Proposer.



1. Let O be the centre of the fixed circle, R its radius; C the centre of the variable circle, and r its radius. Then C is equally likely to fall anywhere within a circle of centre O and radius $R + r$. Let $AB = s$, the arc of the fixed circle covered by the thrown circle. Then the chance that, if a point P be taken at random on the fixed circumference, we shall find $PC > r$, is $p = \frac{s}{2\pi R}$. And if C occupy all its possible positions, the whole chance that $PC < n$ is $p = \frac{M(s)}{2\pi R}$. But if we fix P, the same chance will be $p = \frac{\pi r^2}{\pi (R+r)^2}$, as the favourable positions of C lie within a circle of centre P and radius r. Hence the mean value of the arc AB is $M(s) = 2\pi R \frac{r^2}{(R+r)^2}$. In taking this mean, all the cases are included where the thrown circle does not fall outside the fixed one; so that, when $r < R$, the arc $AB = 0$ when C falls within a circle, centre O, radius $R - r$. Therefore, if, when $r < R$, *we exclude the cases when none of the circumference is covered*, we must multiply the above value of $M(s)$ by $\frac{\pi (R+r)^2}{\pi (R+r)^2 - \pi (R-r)^2}$; therefore $M(s) = \frac{1}{2}\pi r$*.

2. Fix C as before; let the portion of area covered be $AB = \Sigma$; the chance that, if a point P be taken at random within the fixed circle, $PC < r$, is $\frac{\Sigma}{\pi R^2}$. Hence, if C vary, the same chance is $p = \frac{M(\Sigma)}{\pi R^2}$; but, by fixing P, the same chance is $p = \frac{\pi r^2}{\pi (R+r)^2}$; which is always constant; therefore

$$M(\Sigma) = \pi R^2 \frac{r^2}{(R+r)^2}.$$

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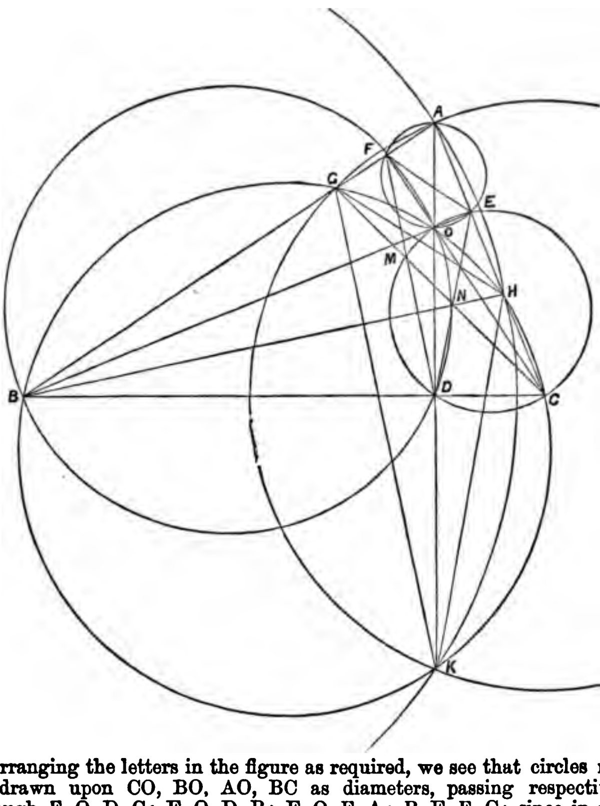
* In Mr. Seitz's solution, when $r > R$, he has omitted the cases when the arc of the fixed circle which is covered is the whole circumference. Also, in part (2), the cases when the area covered is a complete circle are omitted by him. Of course, the cases meant to be omitted are those when the thrown circle falls outside the fixed one.

5684. (By T. Mitchesson, B.A., L.C.P.)—ABC is a triangle; AD, BE, CF are perpendicular to the sides BDC, AEC, AFB; O is the point of intersection of AD, BE, CF; G, H are points in AB, AC such that BH = BA, CG = CA; HK, GK are drawn parallel to ED, FD; CG, DF meet in M; and BH, DE meet in N. Prove that (1) circles may be drawn respectively through A, G, K; A, H, K; B, G, O, H, C, K; O, M, D, C, E; O, N, D, B, F; (2) AD . DO = BD . DC; (3) OM, ON are perpendicular respectively to CG, BH; (4) O is the centre of the circle inscribed in DFE, which triangle (5) is one-fourth of the similar triangle KGH; (6) O is the centre of the circle round AGH; and (7) circles may be drawn to pass through E, O, N, H, and F, O, M, G.

Solution by the Proposer; C. K. Pillai; *and others.*



Arranging the letters in the figure as required, we see that circles may be drawn upon CO, BO, AO, BC as diameters, passing respectively through E, O, D, C; F, O, D, B; F, O, E, A; B, F, E, C; since, in each case, we have two right angles with a common hypotenuse. Now

$\angle$FAO = FEO = DCO = DEO, hence BE bisects $\angle$FED; similarly, it may be shown that CF bisects $\angle$EFD, and AD bisects $\angle$FDE, hence O is the centre of the inscribed circle. As CG = CA, FG = FA, so EH = EA, and $\angle$ACF = ABE = GCF = HBE; but $\angle$ECO = EDO = MDO; therefore M is a point on the circumference EODC; similarly, it may be shewn that N is on FODB. Produce AD, and, if it does not meet GK, HK in K, let it, say, meet HK; then $\angle$OKH = ODE = OBH, therefore a circle passes through B, O, H, K; and, since $\angle$BGC = BOC = BHC, G is a point on the same circle. Now, $\angle$GBO = GKO, clearly K is the point of intersection of AD, HK, GK, since $\angle$GKA = $\frac{1}{2}$ACG; therefore K is a point on the circle, whose centre is C, passing through A, G; so the circle from centre B, passing through A, also passes through K.

And BD . DC = KD . DO = AD . DO, since D is the middle point of AK. The angles OMC, ONB are in semicircles. Because FE = $\frac{1}{2}$GH, and FDE is similar to GKH, the area of FDE is one-fourth of the area of GHK.

Since AF = GF, and OF is perpendicular to AG, AO = GO; similarly AO = HO, therefore O is the centre of the circle described about AGH. And, as $\angle$HEO + $\angle$HNO = 2 right angles, a circle may be drawn round EONH, also one about FOMG.

5727. (By W. H. WALENN.)—Find the remainder to δ of x^m, without using the quotient in the process. As an example, let $\delta = 9$, and

$$x^m - 12648307^{2685461}.$$

Solution by the PROPOSER.

By unitation (see *Phil. Mag.* or 1868, 1873, 1875, 1876, 1877, 1878) the formula for any number (N),

$$a_n r^{n-1} + a_{n-1} r^{n-2} + \dots + a_3 r^2 + a_2 r + a_1{}^n$$

has the same remainder to δ that

$$a_n (r-\delta)^{n-1} + a_{n-1} (r-\delta)^{n-2} + \dots + a_3 (r-\delta)^2 + a_2 (r-\delta) + a_1$$

has. The operation indicated by the latter formula is called unitation, symbolised by U_δN. This operation being performed upon N, and upon the result, until a value less than δ is obtained, constitutes the whole process of unitation. When $\delta = 9$, the coefficient of each term is unity, and the remainder required is $U_9 12648307^{2685461} = U_9 4^5$, since the period of recurrence of the remainders to 9 of the series of powers is 6. Therefore $U_9 x^m$ = the 5th term of the series 471471 ... = 7.

5854. (By Professor WOLSTENHOLME, M.A.) — Perpendiculars are drawn from a fixed point, (1) on three conjugate diameters, (2) on three conjugate diametral planes of a given ellipsoid; prove that in each case

the plane through the feet of the perpendiculars passes through a fixed point, which in (2) lies on the normal drawn at the given fixed point to the ellipsoid through it, similar, concentric, and similarly situate with the given ellipsoid.

Solution by the PROPOSER.

Let (X, Y, Z) be the given fixed point, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ the given quadric; $(l_1 a, m_1 b, n_1 c)$, $(l_2 a, m_2 b, n_2 c)$, $(l_3 a, m_3 b, n_3 c)$ ends of three conjugate diameters, and $p\frac{x}{a}+q\frac{y}{b}+r\frac{z}{c}=1$ the plane through the feet of the perpendiculars on these diameters. Then, where this plane meets the diameter (1),

$$\frac{x}{l_1 a}=\frac{y}{m_1 b}=\frac{z}{n_1 c}=\frac{p\frac{x}{a}+\dots}{pl_1+\dots}=\frac{1}{pl_1+qm_1+rn_1}=\frac{al_1 x+b\dots+\dots}{a^2 l_1^2+\dots},$$

and $al_1(x-\mathrm{X})+bm_1(y-\mathrm{Y})+cn_1(z-\mathrm{Z})=0$, whence

$$(al_1\mathrm{X}+bm_1\mathrm{Y}+cn_1\mathrm{Z})(pl_1+qm_1+rn_1)=a^2 l_1^2+b^2 m_1^2+c^2 n_1^2,$$

and two similar equations. Adding these three equations, and noting that

$$l_1^2+l_2^2+l_3^2=\dots=\dots=1,\text{ and } m_1 n_1+m_2 n_2+m_3 n_3=0,\ \&c.,$$

we get $$ap\mathrm{X}+bq\mathrm{Y}+cr\mathrm{Z}=a^2+b^2+c^2,$$

or the plane $\frac{px}{a}+\frac{qy}{b}+\frac{rz}{c}=1$ passes through the point

$$\frac{x}{a^2\mathrm{X}}=\frac{y}{b^2\mathrm{Y}}=\frac{z}{c^2\mathrm{Z}}=\frac{1}{a^2+b^2+c^2}.$$

Again, the equation of the diametral planes are

$$l_1\frac{x}{a}+m_1\frac{y}{b}+n_1\frac{z}{c}=0,\ \&c.,$$

and the foot of the perpendicular from (X, Y, Z) is (x, y, z), where

$$\frac{x-\mathrm{X}}{\frac{l_1}{a}}=\frac{y-\mathrm{Y}}{\frac{m_1}{b}}=\frac{z-\mathrm{Z}}{\frac{n_1}{c}}=\frac{\frac{l_1}{a}(x-\mathrm{X})+\dots}{\frac{l_1^2}{a^2}+\dots}=\frac{-\left(\frac{l_1\mathrm{X}}{a}+\dots\right)}{\frac{l_1^2}{a^2}+\dots},$$

add also $$=\frac{\frac{p(x-\mathrm{X})}{a}+\dots}{\frac{pl_1}{a^2}+\dots}=\frac{1-\frac{p\mathrm{X}}{a}-\frac{q\mathrm{Y}}{b}-\frac{r\mathrm{Z}}{c}}{\frac{pl_1}{a^2}+\frac{qm_1}{b^2}+\frac{rn_1}{c^2}}.$$

Thus we get $$\left(\frac{l_1\mathrm{X}}{a}+\frac{m_1\mathrm{Y}}{b}+\frac{n_1\mathrm{Z}}{c}\right)\left(\frac{l_1 p}{a^2}+\frac{m_1 q}{b^2}+\frac{n_1 r}{c^2}\right)$$
$$+\left(\frac{l_1^2}{a^2}+\frac{m_1^2}{b^2}+\frac{n_1^2}{c^2}\right)\left(1-\frac{p\mathrm{X}}{a}-\frac{q\mathrm{Y}}{b}-\frac{r\mathrm{Z}}{c}\right)=0,$$

and two like equations; and, on adding and reducing as before,

$$\frac{p\mathrm{X}}{a}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{q\mathrm{Y}}{b}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+r\frac{\mathrm{Z}}{c}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

whence the plane $\frac{px}{a}+\frac{qy}{b}+\frac{rz}{c}=1$ passes always through the fixed point

$$\frac{x}{X\left(\frac{1}{b^2}+\frac{1}{c^2}\right)}=\ldots=\ldots=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},$$

or
$$\frac{x-X}{\frac{X}{a^2}}=\frac{y-Y}{\frac{Y}{b^2}}=\frac{z-Z}{\frac{Z}{c^2}}=\frac{a^2b^2c^2}{b^2c^2+c^2a^2+a^2b^2};$$

so that this fixed point lies on the normal at (X, Y, Z) to the quadric

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}.$$

[This point bisects the distance between (X, Y, Z) and the fixed point through which pass all planes through the ends of three rectangular chords drawn from (X, Y, Z). The corresponding proposition in two dimensions is best proved in the same way.]

5935. (By Donald McAlister, B.A., B.Sc.)—If, in a triangle ABC we have $A \pm B = 90°$, then $2c^{\pm 2} = (a+b)^{\pm 2} + (a-b)^{\pm 2}$, the upper or the lower sign being taken throughout. Required a *single* proof of this extension of Pythagoras' Theorem.

I. *Solution by* Prof. Morel; T. Mitcheson, B.A., L.C.P.; *and others.*

On a, entre les éléments d'un triangle, les relations

$$\frac{a^2}{\sin^2 A}=\frac{b^2}{\sin^2 B}=\frac{c^2}{\sin^2(A+B)}.$$

Si l'on suppose $B = 90° - A$, on en tire $\frac{a^2+b^2}{\sin^2 A+\cos^2 A}=\frac{c^2}{1}$,

ou $c^2=a^2+b^2$; d'où $2c^2=2a^2+2b^2=(a+b)^2+(a-b^2)$.

Si $A = 90°+B$, on a $\frac{a^2}{\cos^2 B}=\frac{b^2}{\sin^2 B}=\frac{c^2}{\cos^2 2B}$,

d'où $a^2-b^2=\frac{c^2}{\cos^2 B-\sin^2 B}=\frac{c^2(\cos^2 B+\sin^2 B)}{\cos^2 B-\sin^2 B}=\frac{(a^2+b^2)c^2}{(a^2-b^2)}$.

On en tire $\frac{2}{c^2}=\frac{2a^2+2b^2}{(a^2-b^2)^2}=\frac{1}{(a+b)^2}+\frac{1}{(a-b)^2}$.

II. *Solution by the* Proposer.

The equation $2c^{\pm 2}=(a+b)^{\pm 2}+(a-b)^{\pm 2}$

may obviously be transformed into

$$2\left\{\sin(A+B)\right\}^{\pm 2}=(\sin A+\sin B)^{\pm 2}(\sin + A-\sin B)^{\pm 2};$$

therefore $$2\left[\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B)\right]^{\pm 2}$$
$$=\left[\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\right]^{\pm 2}+\left[\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)\right]^{\pm 2};$$
$$\therefore 2=\left[\sin\tfrac{1}{2}(A+B)\right]^{\mp 2}\left[\sin\tfrac{1}{2}(A-B)\right]^{\pm 2}+\left[\cos\tfrac{1}{2}(A+B)\right]^{\mp 2}\left[\cos\tfrac{1}{2}(A-B)\right]^{\pm 2},$$
which may be written in the form
$$1=(\surd 2)^{-2}\left[\sin\tfrac{1}{2}(A\pm B)\right]^{\pm 2}\left[\sin\tfrac{1}{2}(A\mp B)\right]^{\pm 2}$$
$$+(\surd 2)^{-2}\left[\cos\tfrac{1}{2}(A\pm B)\right]^{\mp 2}\left[\cos\tfrac{1}{2}(A\mp B)\right]^{\pm 2}.$$
Given, then, that $\frac{1}{2}(A\pm B)=45°$, and taking the upper or the lower sign throughout, the proposed equation reduces to the identity
$$1=(\sin)^2+(\cos)^2.$$
[Mr. McAlister remarks that Professor Morel's solution of his question does not fulfil the imposed condition of *singleness*, inasmuch as the two cases are proved by separate methods.]

1378. (By the late Professor Clifford.)—A tangent to an ellipse is a chord of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters.

Solution by the Proposer.

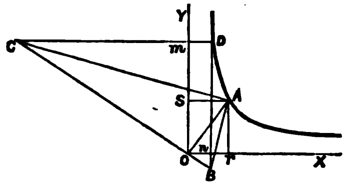
Let the equations to the ellipse, the circle, and the chord, be respectively
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad x^2+y^2=a^2+b^2,\quad \frac{xh}{a^2}+\frac{yk}{b^2}=1 \text{.........}(1,\ 2,\ 3).$$
Then the equation $$x^2+y^2=(a^2+b^2)\left(\frac{xh}{a^2}+\frac{yk}{b^2}\right)^2 \text{ ....}(4)$$
represents the two straight lines passing through the origin and the intersections of (2) and (3). If these are conjugate diameters, we must have
$$-\frac{b^2}{a^2}=\frac{b^4}{a^4}\cdot\frac{a^4-(a^2+b^2)h^2}{b^4-(a^2+b^2)k^2},$$
which may easily be shown to be the case, since $(h,\ k)$ is on the ellipse, and therefore $a^2k^2+b^2h^2=a^2b^2$.

If we change b^2 into $-b^2$, we obtain a similar theorem for the hyperbola but the conjugate diameters will be imaginary, if
$$\frac{h^2}{a^4}+\frac{k^2}{b^4}<\frac{1}{a^2-b}.$$
[This question, with its solution, was received under date May 14, 1863.]

1263. (By the late Professor Clifford, F.R.S.)—Prove that, for every point A on a rectangular hyperbola, there exists a straight line BC, passing

through the centre, such that if, through any other point D on the curve lines be drawn parallel to the asymptotes, cutting BC in B, C, the intercept BC subtends a right angle at A.

I. *Solution by* STEPHEN WATSON.

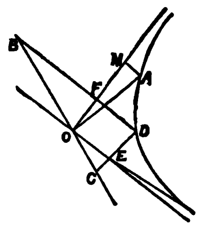


Let O be the centre, OX, OY the asymptotes, and A a point on the curve. Join OA, and through O draw BC perpendicular to OA; draw Dm, Dn parallel to OX, OY, meeting BC in C and B; also As, Ar parallel to the same, and join AB, AC. Then, by similar triangles, we have $\quad OA : Ar = OB : On$, and $OA : As = Oc : Om$,

therefore $\quad OA^2 : Ar \,.\, As = OB \,.\, OC : Om \,.\, On.$

But $Ar \,.\, As = Om \,.\, On$; therefore $OA^2 = OB \,.\, OC$; and therefore, since AOB is a right angle, BAC is also a right angle.

II. *Solution by the* PROPOSER.



Let O be the centre of the hyperbola; join OA, and draw BOC perpendicular to it. Let DB, DC, parallel to the asymptotes, cut them in F, E; and draw AM perpendicular to either of the asymptotes.

Then we have $CO : OB = CE : E'D$,

$\therefore \quad CO^2 : CO \,.\, OB = OE \,.\, EC : OE \,.\, ED$......(1).

But, by similar triangles OCE, OAM,

$CO^2 : OE \,.\, EC = OA^2 : OM \,.\, MA$........ (2);

therefore, comparing (1) and (2), since

$OE \,.\, ED = OM \,.\, MA, \quad OA^2 = CO \,.\, OB,$ or BAC is a right angle.

The property holds for any hyperbola, but the line BC does not always pass through the centre; if it cut the asymptotes in P and Q, the angles APQ, AQP are each equal to the angle between the asymptotes, and to BAC. I have not been able to find a geometrical construction for the line.

[This Solution bore date March 30, 1863. In a subsequent theorem, whereof solutions have been published in our volumes of *Reprints*, Mr. CLIFFORD gave the generalization above referred to for any hyperbola.]

5960. (By Professor WOLSTENHOLME, M.A.)—On the normal to an ellipse at P is measured (outwards) PO equal to the radius of curvature, and the normal meets the director circle in Q, Q′; prove that (QOPQ′) is an harmonic range.

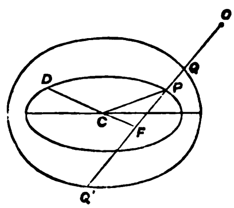
I. *Solution by* D. EDWARDES; R. GRAHAM, M.A.; *and others.*

If CF be perpendicular upon the normal from the centre of the ellipse, we have $PQ \cdot PQ' = QE^2 - PF^2 = CP^2 + CD^2 - CF^2 - PF^2 = CD^2$......(1);
also $QE^2 = CQ^2 - CE^2 = CP^2 + CD^2 - CF^2 = CD^2 + PF^2$,
that is, $(PQ + PQ')^2 = 4CD^2 + 4PF^2$, or $PQ' - PQ = 2PF$ by (1)......(2).

Now the radius of curvature $= \dfrac{CD^2}{PE}$;

hence $$\frac{1}{PQ} - \frac{1}{PQ'} = \frac{1}{OP}.$$

II. *Solution by the* PROPOSER.



If F be the middle point of QQ', we have
$$FQ^2 = FP^2 + QP \cdot PQ' = FP^2 + CD^2,$$
for $$QP \cdot PQ' = CQ^2 - CP^2 = AC^2 + BC^2 - CP^2 = CD^2,$$
and $$FP \cdot FO = FP\left(FP + \frac{CD^2}{F}\right) = FP^2 + CD^2 = FQ^2,$$
that is, $FP : FQ = Q'F : FO$;
$$FQ - FP : FQ + FP = FO - FQ : Q'F + FO,$$
$$PQ : Q'P = QO : Q'O, \text{ or } -1 = \frac{PQ' \cdot QO}{PQ \cdot Q'O} = \frac{QO \cdot PQ'}{Q'O \cdot PQ}.$$

III. *Solution by* J. HAMMOND, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Using (x, y) to denote the coordinates of either of the points Q, Q', and R to denote its distance from P, we have
$$(x - a\cos\phi)\, a\sin\phi = (y - b\sin\phi)\, b\cos\phi, \quad x^2 + y^2 = a^2 + b^2 \ldots\ldots (1,\ 2),$$
$$R = \left\{(x - a\cos\phi)^2 + (y - b\sin\phi)^2\right\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we readily obtain
$$x = a\cos\phi + \frac{bR\cos\phi}{b'}, \quad y = b\sin\phi + \frac{aR\sin\phi}{b'}.$$

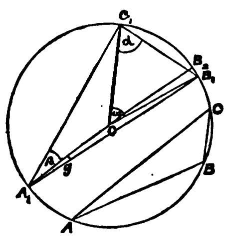
Substituting these values in (2) [and remembering that
$$b^2\cos^2\phi + a^2\sin^2\phi = b'^2,]$$
we get $R^2 + \dfrac{2ab}{b'}R - b'^2 = 0$, an equation whose roots are PQ and PQ'.

Hence, clearly, $\dfrac{2PQ \cdot PQ'}{PQ + PQ'} = \dfrac{b'^3}{ab}$ = radius of curvature at P; and the theorem in the question follows.

5908. (By Professor DARBOUX.)—Dans une circonférence donnée O, on inscrit un triangle donné quelconque ABC, qu'on suppose décrit dans le sens ABC. Parallèlement à ses côtés et dans leurs sens respectifs, on trace les rayons OA_1, OB_1, OC_1, et l'on forme le triangle $A_1B_1C_1$. On agit sur le deuxième triangle comme sur le premier, et ainsi de suite indéfiniment. On demande (1) vers quelle forme tendent ces triangles; (2) s'il y a des lignes déterminées dont s'approchent indéfiniment les côtés, et, lorsqu'il y en a, de les faire connaître ; (3) si, en remontant par l'opération inverse la série des triangles $A_1B_1C_1$, ABC, $A_{-1}B_{-1}C_{-1}$, etc., où l'on regarde toujours les éléments de ABC comme donnés, l'opération pourra se poursuivre indéfiniment, et, si l'on doit s'arrêter, pour quel rang cette circonstance se présentera.

Solution by PRINCE DE POLIGNAC.

From the law of construction, OB_1 is parallel to AB, OC_1 to BC, OA_1 to CA. Putting $\angle B_1OC_1 = \omega$, we have $\omega = \pi - B$; and each of the angles A_1, &c., is the sum of two other angles, viz., $A_1 = \beta+\gamma$, $B_1 = \gamma+\alpha$, $C_1 = \alpha+\beta$; moreover $2\alpha+\omega = \pi$, therefore $\alpha = \frac12 B$, $\beta = \frac12 C$, $\gamma = \frac12 A$. We have then



$$A_1 = \tfrac12(C+A),\ B_1 = \tfrac12(A+B),\ C_1 = \tfrac12(B+C),$$

or
$$A_1 = \tfrac12\pi - \tfrac12 B,\ B_1 = \tfrac12\pi - \tfrac12 C,$$
$$C_1 = \tfrac12\pi - \tfrac12 A,$$

therefore

$$A_2 = \tfrac12\pi - \tfrac12 B_1, \text{ or } A_2 = \tfrac12\pi - \tfrac12(\tfrac12\pi - \tfrac12 C);$$

$$A_2 = \frac{2-1}{2^2}\pi + \frac{C}{2^2};\ B_2 = \&c.$$

The formula for A_2 can be written $A_2 = \dfrac{2-1}{2^2}\pi + (-1)^2\dfrac{(A)^2}{2^2}$, where $(A)^2$ stands for the second power of the circular substitution (ABC), and under that form it is easily generalized and leads to

$$A_n = \frac{2^{n-1}-2^{n-2}+2^{n-3}\dots\pm 1}{2^n}\pi + (-1)^n\frac{(A)^n}{2^n},$$

where A can be replaced by B or C. Reducing, we obtain the general formula $\quad A_n = [1+(-1)^{n-1}2^{-n}]\,\frac13\pi + (-1)^n 2^{-n}.(A)^n$ ....(1), which gives the angles of the successive triangles obtained through the direct graphic process, or the law of magnitude.

As regards the law of displacement, we have OB_2 parallel to A_1B_1, hence $\angle B_1OB_2 = \angle OB_1A_1 = \gamma = \frac12 A$. Hence $\angle B_2OB_3 = \frac12 A_1$, and generally $\quad 2B_1OB_n = A + A_1 + \dots A_{n-2}$..........................(2).

Finally, it will easily be seen that we need only change n to $-n$ in formula (1) in order to get the law of magnitude for the angles in the inverted graphic process. Thus we have

$$A_{-n} = [1+(-1)^{n+1}2^n]\,\tfrac13\pi + (-1)^n 2^n (A)^{-n} \text{ ....(3).}$$

We can now answer the three parts of the question.

1. When n increases indefinitely in formula (1), A_n converges towards $\frac13\pi$. Hence, in the direct process, the ultimate triangle is equilateral. Observe that, if we start with an equilateral triangle, we obtain another equilateral triangle.

2. We have to find the limit of $\angle B_1OB_n$. The sum of any six consecutive terms in formula (2) can be expressed through formula (1), and will be found to be, noting that $(A)^{n+3} = (A)^n$, and that $(A)^n + (A)^{n+1} + (A)^{n+2} = \pi$, $\quad 2\pi + (-1)^n 2^{-(n+5)} . 21 [(A)^n - (A)^{n+1}]$.

Rejecting 2π, we have $2B_1OB_{n+2} = \Sigma(-1)^n 2^{-(n+5)} . 21 [(A)^n - (A)^{n+1}]$, with $n = 0, 6, \dots 6m$, that is, as $(A)^{6m} = A$, and $(A)^{6m+1} = B$,

$$\lim B_1OB_n = 21(A-B)\sum_{n=0}^{n=\infty} 2^{-(n+6)} \text{ or } \lim B_1OB_n = 2^{-6} . 21(A-B)\Sigma 2^{-n};$$

$$(n = 6m) \lim B_1OB_n = 2^{-6} . 21(A-B) \text{ and } \frac{1}{1-2^{-6}} = \tfrac{1}{3}(A-B),$$

whereby the position of the ultimate triangle is determined. If we start with an equilateral triangle, we have $2B_1OB_2 = A = \frac{1}{3}\pi$, $B_1OB_2 = \frac{1}{6}\pi$, and the vertices of the successive triangles coincide with those of an inscribed dodecagon.

3. Making $n = 1$ in (3), we get $A_{-1} = \pi - 2(A)^{-1}$ or $A_{-1} = \pi - 2C$. Hence the inverted process must stop when we get to a triangle which has an angle $C \overline{\gt} \frac{1}{2}\pi$, as the next triangle would then have an angle $\overline{\lt} 0$. In fact, as A_{-n} is to be > 0, formula (3) gives, when n is even, an inferior limit for $(A)^{-n}$, in other words for each of the angles A, B, C of the original triangle, and a superior limit when n is odd. We have

$$n = 2m, \ (A)^{-2m} > \left(1 - \frac{1}{2^{2m}}\right)\tfrac{1}{3}\pi, \quad n = 2m+1 \ (A)^{-(2m+1)} < \left(1 + \frac{1}{2^{2m+1}}\right)\tfrac{1}{3}\pi.$$

For $m = \infty$ both limits coincide and are $= \frac{1}{3}\pi$. The rank n of the last triangle in the negative series is in every particular case inferred from the conditions

$$\left(1 - \frac{1}{2^{m}}\right)\tfrac{1}{3}\pi < A, B, C, < \left(1 + \frac{1}{2^{2m+1}}\right)\tfrac{1}{3}\pi.$$

5927. (By Professor CAYLEY.)—If $\{\alpha+\beta+\gamma\dots\}^p$, denote the expansion of $(\alpha+\beta+\gamma\dots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\delta^d\dots$ only in which $b+c+d\dots \not> p-1$, $c+d\dots \not> p-2$, &c. &c.; prove that

$$x^n = (x+\alpha)^n - (n)_1 \{\ \}^1 (x+\alpha+\beta)^{n-1} + \frac{n(n-1)}{1.2}\{\alpha+\beta\}^2 (x+\alpha+\beta+\gamma)^{n-2}$$
$$- \frac{n(n-1)(n-2)}{1.2.3}\{\alpha+\beta+\gamma\}^3 (x+\alpha+\beta+\gamma+\delta)^{n-3} + \&c.\dots(1).$$

Solution by H. STABENOW, M.A.

Let us assume the proposed equation to be identically true for some positive integer n, and consequently (as follows from differentiation with respect to x) true also for any of the consecutive numbers 1, 2, ... $(n-1)$; writing $\alpha_1, \alpha_2, \alpha_3$, &c. for α, β, γ, &c.

If now, firstly, we multiply the different terms of (1), from left to right respectively, by

x, $[(x+a_1)-a_1]$, ..., $[(x+a_1+a_2+\ldots+a_{n+1})-(a_1+a_2+\ldots+a_{n+1})]$, and in the result thus obtained substitute for $(x+a_1)^n$, $(x+a_1+a_2)^{n-1}$, ... their values deduced from (1), treating $x+a_1$, $x+a_1+a_2$, ... as single quantities, we plainly reduce it, since in it the signs are alternately + and −, to this form

$$x^{n+1} = (x+a_1)^{n+1} - (n+1)_1 a_1 (x+a_1+a_2)^n + A_2 (x+a_1+a_2+a_3)^{n-1} - \ldots$$
$$\ldots \pm A_p (x+a_1+\ldots+a_{p+1})^{n+1-p} \mp \ldots \pm A_{n+1} \ldots\ldots\ldots\ldots (2).$$

And if, secondly, we multiply both members of (1) by dx, integrate and determine the constant C by making $x=0$, we get

$$x^{n+1} = (x+a_1)^{n+1} - (n+1)_1 a_1 (x+a_1+a_2)^n$$
$$+ (n+1)_2 \{a_1+a_2\}^2 (x+a_1+a_2+a_3)^{n-1} - \ldots$$
$$\ldots \pm (n+1)_p \{a_1+\ldots+a_p\}^p (x+a_1+\ldots+a_{p+1})^{n+1-p} \mp \ldots \pm C \ldots (3),$$

where in (2) the value of any of the coefficients, as A_p is
= the p^{th} coefficient in (1) + one term accruing to it from each of the expansions of the p quantities $(x+a_1)^n$, $(x+a_1+a_2)^{n-1}$, ...

$$= (n)_p \{a_1+a_2+\ldots+a_p\}^p + (n)_{p-1} \times$$
$$\left\{\begin{array}{l} \{a_1+\ldots a_{p-1}\}^{p-1}(a_1+\ldots a_p) + (p-1)_1\{a_1+\ldots a_{p-2}\}^{p-2}(a_1+\ldots a_{p-1})\,a_p + \ldots \\ + (p-1)_2\{a_1+a_2\}^2 (a_1+a_2+a_3)\{a_4+a_5+\ldots+a_p\}^{p-3} \\ \qquad + (p-1)_1 a_1 (a_1+a_2)\{a_3+\ldots+a_p\}^{p-2} + a_1\{a_2+a_3+\ldots+a_p\}^{p-1} \end{array}\right\},$$
$$\ldots\ldots\ldots\ldots (4)_a,$$

or, designating the aggregate of terms that multiply $(n)_{p-1}$ by B_{p-1},

$$= (n)_p \{a_1+a_2+\ldots+a_p\}^p + (n)_{p-1} B_{p-1}.$$

But evidently equations (2) and (3) must be satisfied by the same systems of arbitrarily chosen values for x, a_1, a_2, ... a_{n+1}, which is impossible, unless any term in one be equal to the corresponding term in the other; therefore, generally,

$$(n+1)_p \{a_1+\ldots+a_p\}^p = [(n)_p + (n)_{p-1}]\{a_1+\ldots+a_p\}^p$$
$$= (n)_p \{a_1+\ldots+a_p\}^p + (n)_{p-1} B_{p-1},$$

and
$$\{a_1+\ldots+a_p\}^p = B_{p-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)_b.$$

This last relation, always co-existing with (1) for $p = 1, 2, 3 \ldots n$, must exist, too, for $p = (n+1)$; for otherwise, by what has just been shown, we must have $x^{n+1} \lesseqgtr (x+a_1)^{n+1} - (n+1)_1 a_1 (x+a_1+a_2)^n +$ &c.;

and, differentiating this inequality,

$$x^n \gtrless (x+a_1)^n - (n)_1 a_1 (x+a_1+a_2)^{n-1} + \text{\&c.},$$

a result incompatible with our first hypothesis.

In virtue of the relations $(4)_a$ and $(4)_b$, equation (2) becomes obviously, by substituting the values of $A_2, A_3, \ldots A_{n+1}$,

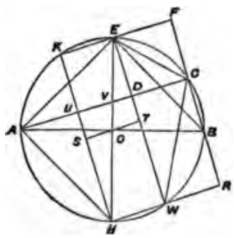
$$x^{n+1} = (x+a_1)^{n+1} - (n+1)_1 a_1 (x+a_1+a_2)^n$$
$$+ (n+1)_2 \{a_1+a_2\}^2 (x+a_1+a_2+a_3)^{n-1} - \ldots,$$

which, being of precisely the same form as (1), proves that this latter, if true for n, will be true for $(n+1)$; but it is true for $n=1$, $n=2$.

Hence it is true for any positive integer n.

5857. (By the Editor.)—A square is constructed on the hypotenuse of a right-angled triangle; prove that the distance of the middle of this square from each of the sides that contain the right angle of the triangle is equal to half the *difference*, or to half the *sum*, of these sides, according as the triangle and the square are on the *same* or on *opposite* sides of the hypotenuse.

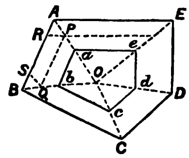
Solution by the Proposer.



Let ABC be the triangle, whereof AB is the hypotenuse and AC, CB the sides. Draw the circumscribed circle AECBH; the vertical diameter EH, cutting AC in V; and the perpendiculars EDW, EF, HGK upon AC, CB. Then E, H are the middles of the squares on the hypotenuse. Now, by comparison of the two pairs of identically equal triangles (ABC, EHW), (AHV, CWD), we have at once UD = BC and AU = CD, whence it follows, immediately, that HR = UC = half the *sum* of AC, CB; EF = DC = half the *difference* of AC, CB.

6033. (By E. H. Moore.)—Out of paper, or anything suited to the purpose, cut any rectilinear figure you please, having five unequal sides; then cut a second similar figure, either greater or less than the first. You will thus have two irregular pentagons. similar in shape, but unequal in dimensions, which you may cut, in any way you please, into any number of compartments you think fit for the solution of the problem. Now the problem consists in forming and arranging these compartments, so as to construct out of them, or by means of their combinations when placed side by side, four other pentagonal figures, similar to the two originally assumed, and also subject to the restriction, that of the four figures so constructed two must be of one size, and two of another size.

I. *Solution by* J. E. A. Steggall, M.A.; J. O'Regan; *and others.*



Arrange the pentagons with their similar sides parallel, and so that a similar point O in each coincides: let the figures be ABCDE and *abcde*; then bisect A*a*, B*b* by PQ, and draw PR parallel to BO, QS parallel to AO, meeting AB in RS. Continue this all round, then, clearly, piece PQRS = piece PQ*ba*, and therefore, when added on to (*abcde*), gives a figure equal to PQ ..., &c., and similar to the given ones. Also, we have left 10 pieces, APR, SQB, &c., which are similar to O*ab*, &c.: these then will form *two* polygons similar to the other, and whose sides are to that of larger as AR : AB.

Also, this theorem is perfectly general and holds good for n-sided figures.

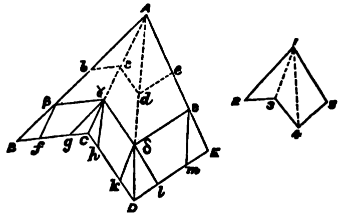
Again, let $AB = a$, $ab = b$, then clearly $AR = \frac{1}{2}(a-b)$, $PQ = \frac{1}{2}(a+b)$,

and $$2\left[\tfrac{1}{2}(a-b)\right]^2 + 2\left[\tfrac{1}{2}(a+b)\right]^2 = a^2+b^2,$$

which gives an analytical form to our result.

II. *Solution by* E. B. Elliott, M.A.

A solution of this puzzle is suggested to me by the following theorem of my own, given in the *Messenger of Mathematics* for February, 1878:—that if the two ends A, B of an elastic string kept constantly stretched move simultaneously all round the perimeters of two closed areas (A), (B), and if (P) be the area surrounded by the locus of the point dividing AB in the fixed ratio $m : n$, and S the area described by A relatively to B, then



$$(P) = \frac{m(B) + n(A)}{m+n} - \frac{mn}{(m+n)^2}S.$$

In the particular case of this where $m = n$, *i.e.*, where (P) is surrounded by the path of the bisecting point, this becomes

$$(P) + \tfrac{1}{4}S = \tfrac{1}{2}\left\{(A) + (B)\right\}.$$

Let now (A) be the area of the larger, ABCDE of the two given similar pentagons, (B) that of A*bcde*, which is equal and similar to the smaller 12345, but has the vertex homologous to A at A, and the sides A*b*, A*c* homologous to AB, AC along those sides respectively; and let the one end of the string move from B to C, from C to D, from D to E, from E to A, and from A to B, while the other does from *b* to *c*, from *c* to *d*, from *d* to *e*, from *e* to *a*, and from *a* to *b*; and we get, by applying the above, two areas (P) and $\frac{1}{4}$S, the sums of two of each of which make up the sum of (A) and (B). It is also clear, from the way of describing, that the two areas (P) and $\frac{1}{4}$S are rectilinear, and similar to (A) and (B), as the question necessitates.

But all may be easily seen by dissection from first principles, as follows:—

Bisect bB, cC, dD, eE in β, γ, δ, ϵ respectively. Join $\beta\gamma$, $\gamma\delta$, $\delta\epsilon$. Draw βf parallel to cC, and γg parallel to $b\beta$, meeting BC in m and n; γh parallel to dD, and δk parallel to cC, meeting CD in h and k; and δl parallel to eE, and ϵm parallel to dD, meeting DE in l and m; then

$$\Delta Abc = \Delta A\beta\gamma - b\beta\gamma c,$$

and $\quad \Delta ABC = \Delta A\beta\gamma + g\gamma\beta f + 2\Delta\beta Bf = \Delta A\beta\gamma + bB\gamma c + 2\Delta\beta Bf;$

therefore $\quad \Delta Abc$ and $ABC = 2\Delta A\beta\gamma$ and βBf,

that is, $\quad \Delta 123$ and $ABC = 2\Delta A\beta\gamma$ and βBf.

So $\quad \Delta 134$ and $ACD = 2\Delta A\gamma\delta$ and γCh,

and $\quad \Delta 145$ and $ADE = 2\Delta A\delta\epsilon$ and δDl.

Therefore, adding, pentagon 12345 + pentagon ABCDE

= twice pentagon $A\beta\gamma\delta\epsilon$ + twice the pentagon made up by Δ βBf, γCh, δDl,

which last pentagon is of course also equal to the one made up of $\Delta\ \gamma g C$ $\delta k D$, $\epsilon m E$.

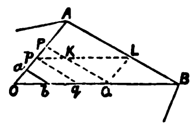
Thus the dissections are along the full lines in the figures; and the rearrangements are $A\beta\gamma\delta\epsilon$ for one polygon; 12345, with $g\gamma\beta f$, $k\delta\gamma h$, and $m\epsilon\delta l$, for a second equal one; $\beta B f$, $\gamma C h$, and $\delta D l$ for a third; and $\gamma g C$, $\delta k D$, and $\epsilon m E$ for a fourth equal to it.

Note 1.—Though stated for pentagons only, the state of things is just the same for two hexagons, heptagons, or any similar polygons whatever.

Note 2.—The above is the simplest dissection, such as wanted, but is, of course, only one of an infinite number of such possible dissections. For, instead of being placed with a vertex A and two sides common, the two polygons might have been put anyhow homothetically, and the division of them been similarly into triangles radiating from their then centre of similitude.

III. *Solution by the* Rev. F. D. Thomson, M.A.

Let any two similar polygons ABC..., *abc*... be placed one within the other, so that the homologous sides AB, *ab*, &c. are parallel. Then A*a*, B*b*, C*c*, ... will meet in a point O.



Bisect *a*A in P, *b*B in Q, &c., and take $Op = aP$, $Oq = bQ$, &c. Then PQ..., *pq*... will be two polygons similar and similarly situated to the original two.

Draw PKL parallel to OB, and join LQ.

Then, since $pP = OP - Op = OP - aP = Oa$, the triangle Ppk is equal in all respects to the triangle aOb, and the triangle ApL to the triangle POQ. Also, each of the triangles BQL, QLB is equal in all respects to the triangle pOq.

Therefore $AOB = POQ + APKL + 2pOq = 2POQ + 2pOq - aOb$;

therefore $AOB + aOb = 2POQ + 2pOq$.

Hence, making a similar construction for the other triangles having their vertex at O and making up the polygons, we see that the pieces corresponding to APKL, POQ, KQL, LQB will, with the pieces corresponding to aOb, make up two polygons equal to POQ, and two polygons equal to pOq.

Hence there is an infinite number of ways in which the problem can be solved, since the point O may be taken as convenient.

This solution was suggested by the fact that

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2,$$

and that the areas of similar figures vary as the squares of their homologous sides.

5566. (By Professor Minchin, M.A.)—1. A force being completely represented by a vector ω at the extremity of a vector α drawn from an arbitrary origin, prove that the moment of the force about an axis drawn at the extremity of a vector β in the direction of the unit vector θ is $S(\alpha - \beta)\,\omega\theta$, and show how from this expression the discussion of Poinsot's and other axes of principal moment proceeds.

2. In particular, prove very simply that the axes of principal moment at points situated on any right line trace out a hyperbolic paraboloid.

Solution by J. J. WALKER, M.A.

The moment in question is equal numerically to the volume of the parallelepiped, the three edges of which are $\alpha-\beta$, ω, θ, since θ is a unit vector. And $\theta S\,\theta(\alpha-\beta)\,\omega = \theta S\,.\,\theta V(\alpha-\beta)\,\omega = V\,.\,V\omega\theta V\theta(\alpha-\beta)$, the expression for the vector moment given in Quest. 5912, as shown in the Solution, *Educational Times*, May 1, p. 151. I may mention that I had overlooked the present Quest. in proposing 5912, the expression given in which I had arrived at quite otherwise than as a transformation $\theta S\,\theta(\alpha-\beta)\,\omega$. Thus, for a system of forces, $S\theta\Sigma(\alpha-\beta)\,\omega$ will plainly be a maximum about an axis parallel to $V\Sigma(\alpha-\beta)\,\omega$, and the moment about any other axis will be the projection of the principal moment on that axis. The equation of Poinsot's axis cannot be more simply and directly investigated than as in Tait's *Quaternions*, p. 223.

To prove the more interesting part of this Question. Suppose the origin on the given line $x\gamma$. The axis of principal moment at any point on it is $\sigma = x\gamma + yV\Sigma(\alpha - x\gamma)\,\omega$, the locus of which is a line parallel to the plane of the two fixed vectors $V\Sigma\alpha\omega$, $V\gamma\Sigma\omega$. But since, too,

$$\sigma = yV\Sigma\alpha\omega + x(\gamma - yV\gamma\Sigma\omega),$$

the locus is also a variable line resting on $V\Sigma\alpha\omega$ and parallel to the plane of the vectors γ, $V\gamma\Sigma\omega$; and plainly every generator of one system intersects every generator of the other. Hence the axes of resultant moment for points situated in a given line generate a hyperbolic paraboloid, the Cartesian equation to which, referred to γ, $V\Sigma\alpha\omega$, and $V\Sigma\omega\gamma$, is $pxy = qz$.

5971. (By DONALD MCALISTER, B.A., B.Sc.)—If n be an odd number, prove that its reciprocal is equal to the series

$$\frac{n}{2\,!}\,.\,2 - \frac{(n+1)\,n\,(n-1)}{4\,!}\,.\,2^2 + \frac{(n+2)(n+1)\,n\,(n-1)\,(n-2)}{6\,!}\,2^3 - \ldots + \frac{1}{2n}\,.\,2^n.$$

Solution by D. EDWARDES; G. TURRIFF, M.A.; *and others.*

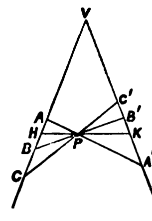
$$\tfrac{1}{2}\log\frac{1+x^2}{(1-x)^2} = \tfrac{1}{2}\log\left\{1 + \frac{2x}{(1-x)^2}\right\} = \frac{x}{(1-x)^2} - \frac{2x^2}{2\,(1-x)^4} + \frac{2^2\,.\,x^3}{3\,(1-x)^6}\ldots\text{(A)}.$$

The coefficient of x^n in the expansion of $\dfrac{x^p}{(1-x)^{2p}}$ is easily found to be $\dfrac{n\,[n^2-1^2]\ldots[n^2-(p-1)^2]}{(2p-1)\,!}$. Hence, equating the coefficients of x^n on both sides of A, we get (if n be odd) the result in the Question.

5890. (By the Rev. G. H. HOPKINS, M.A.)—A point P is taken in the same plane as the two straight lines ABC..., and A′B′C′...; any

number of straight lines AA′, BB′, CC′, ... are drawn through the point P, and terminate on the given lines. Upon these, as diameters, circles are drawn with their planes at right angles to the plane AA′BB′... ; if tangents be drawn to these circles at the points where they are met by the common perpendicular line from P, these tangents will meet the plane AA′BB′... in a straight line.

Solution by the PROPOSER.



Let the straight lines meet in V, and draw HPK making equal angles with them; with vertex V, and base the circle which has HK as diameter, and plane perpendicular to the plane AA′BB′..., describe a cone; let Q be the point on the cone where the line from P and perpendicular to the plane AA′BB′... meets it; then it is obvious that the tangent plane to VQ will touch all the elliptical sections AQA′, BQB′... at the common point Q, and cut the plane AA′BB′... in a straight line, and this is the line where all the tangents to the elliptical sections at the common point Q will meet the plane, and the tangents to the points on the auxiliary circles which correspond to Q will meet the plane in the same line, and these auxiliary circles are the same as those mentioned in the theorem.

The position of this line is easily obtained by drawing CPC′ so that it is bisected in P, then the line through V and parallel to CPC′ is the line required.

In Plane Geometry, this theorem becomes: If TPT′ be drawn through P to meet the circle described on APA′ as diameter, and at right angles to APA′, then the tangents at T and T′ will intersect in a line, whatever be the position of APA′.

6004. (By EDWYN ANTHONY, M.A.)—If θ, θ' are the excentric angles corresponding to the extremities of any focal chord of an ellipse whose excentricity is e, prove that $\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta' = \dfrac{e-1}{e+1}$.

Solution by R. KNOWLES, L.C.P.; T. R. TERRY, M.A.; *and others.*

Since the line joining the points passes through the point $(ae, 0)$,

we have $$\frac{b\sin\theta}{a\cos\theta - ae} = \frac{b(\sin\theta' - \sin\theta)}{a(\cos\theta' - \cos\theta)},$$

therefore $$e = \frac{\sin(\theta-\theta')}{\sin\theta - \sin\theta'} = \frac{\cos\frac{1}{2}(\theta-\theta')}{\cos\frac{1}{2}(\theta+\theta')},$$

therefore $$\frac{e-1}{e+1} = \frac{\cos\frac{1}{2}(\theta-\theta') - \cos\frac{1}{2}(\theta+\theta')}{\cos\frac{1}{2}(\theta-\theta') + \cos\frac{1}{2}(\theta+\theta')} = \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'.$$

$d\omega$ is the area of the element, r the radius of the bubble,

$$2\pi\sigma^2 d\omega = 2\frac{\mathrm{T}}{r}\,d\omega, \quad \text{therefore } \pi\sigma^2 = \frac{\mathrm{T}}{r}.$$

Now, if Q is the whole charge, $\mathrm{V} = \frac{\mathrm{Q}}{r} = 4\pi r\sigma$; hence $\mathrm{V}^2 = 8\pi r\mathrm{T}$.

On Traditional Testimony. *By* C. J. Monro, M.A.

I had missed, until recently, the discussion in 1877 (*Reprint*, Vol. XXVII, pp. 77—79, 89—91), but Dr. Macfarlane's paper (*Reprint*, Vol. XXXII., p. 18) seems to show that the subject is not considered to be exhausted. After Prof. Cayley's paper in 1877, I suppose it was admitted by all parties, except the simple-product party, whose principles I do not understand, that if an answer is to be given in terms of p_1 and p_2 only, nothing more could be said than that the probability lies between $p_1 p_2$ and $p_1 p_2 + 1 - p_1$ or $1 - p_1(1 - p_2)$. But Dr. Macfarlane brings out what I dare say others besides myself had overlooked, that, thus considered, the question is formally identical with that of a syllogism in the first figure with probable premises; Minor, If A says that B said, &c., B did say, &c.; Major, If B said that E happened, E did happen; Conclusion, If A says that B said that E happened, E did happen. (Boole's *Laws of Thought*, p. 285.)

But such a solution underrates the insufficiency of the data. The data do not import that the witnesses were virtually interrogated, that is, that they could not have been silent if they chose; and this consideration is not superseded by evidence that in fact they spoke. Accordingly, Prof. Cayley, while supplying part of the insufficiency of the data, makes further reserves by expressly assuming "that the event is of such a nature that, when there is *any* testimony in regard to it, the probability is determined by that testimony irrespectively of the antecedent probability;" and I suppose this assumption as to the event which B is said to have reported applies also to the event of his reporting it, because p_1 is taken as the probability that this event happened. It would be a valuable contribution to the question if Prof. Cayley would explain—what I do not understand—how the nature of such events is to be defined. I understand an allegation which, to us, has no antecedent probability, or let us say no *fore-chance*, at all; at any rate we have one when we do not know what the allegation is; for instance, in a statement in Talleyrand's *Memoirs*. But Prof. Cayley supposes an event with a definite fore-chance k, which appears only in the term depending on the hypothesis, not excluded by the data, that B gave no information. With a strong sense of being liable to correction, I venture to think that, if events of the nature in question are logically conceivable, they must be mathematically definable, and to suggest that if, in dealing with them, we begin by taking the fore-chance into account, it ought to disappear from a correct expression for the *after-chance*. There is indeed a well-known class of testimony cases in which the fore-chance does thus disappear; namely, when a single witness reports one of a number of events equally likely to happen, and equally likely to be wrongly reported if they did not happen, as in the first case considered by Laplace in the

of every such sphere, $r_1 r_2 = 2rz$, where r is the radius of the sphere, we get, after some trifling reductions,

$$\frac{dV}{dx} = \frac{m}{(r_1 r_2)^3} \cdot x, \quad \frac{dV}{dz} = \frac{m}{(r_1 r_2)^3} \left[z + r\left(\sec^2 \tfrac{1}{2}\phi - \cos\phi\right)\right],$$

which manifestly prove the two properties in question.

The fixed point through which, as shown by the preceding values of its components parallel and perpendicular to the plane of the lamina, the resultant attraction passes, for each segment of the sphere, is easily seen to be that harmonically conjugate to the centre of the sphere with respect to the segment of the axis of the lamina intercepted between the pole of the opposite segment and the vertex of the cone enveloping the sphere along the circumference of the lamina.

5338. (By S. Roberts, M.A.)—If two surfaces of the degrees m, n have a common isolated right line and are otherwise general, show that they touch one another at $m+n-2$ points on the line.

Solution by the Proposer.

For the surfaces are given by $xu + yv = 0$, $x\phi + y\chi = 0$, where u, v are functions of the $m-1^{\text{th}}$ order and ϕ, χ are of the $n-1^{\text{th}}$ order.

The tangent planes at points on the line are

$$x(u) + y(v) = 0, \quad x(\phi) + y(\chi) = 0,$$

where $x = 0$, $y = 0$ are substitutes for x, y in u, &c., which are then denoted by (u), &c. Hence, if the planes coincide, we have

$$(u)(\chi) - (v)(\phi) = 0,$$

containing z in the power $m+n-2$ which is the number of coincident tangent planes.

6055. (By H. L. Orchard, L.C.P.)—A spherical soap-bubble is electrified in such a manner that it is just in equilibrium when the pressures of the external and internal air are equal. Show that if V be now its potential, V varies as $T^{\frac{1}{2}}$, where T is the tension of the film across a line of unit length.

Solution by Professor Minchin, M.A.

If σ is the density of the electrification, the intensity of repulsion at any point of the surface is $2\pi\sigma^2$. There being no excess of air pressure, the equilibrium of a superficial element is expressed by equating the electrical repulsion on it to the normal component of the surface-tension. Thus, if

which agrees with Laplace's general solution $\dfrac{n\,P_x - 1}{n-1} = \Pi_x \dfrac{np-1}{n-1}$, where P_x is the chance given by a tradition of x links and Π_x denotes a product of x factors varying by p.

The unmodified solution (1) reduces itself to that quoted from TODHUNTER's Algebra if (i) $\theta = 1$, or B virtually interrogated; (ii) $\theta' = 1$, or further, only two possible events in question, and B equally likely to speak the truth in either event; (iii.) $a' = a$, or further, A equally likely to speak the truth in either event; and (iv.) $k = \frac{1}{2}$, the two events equally likely beforehand. Part of these conditions is indicated by LACROIX (ed. 2, 1822, § 142) in the phrase "temoignages rendus par oui ou par non." He follows PREVOST and L'HUILLIER; but Mr. TODHUNTER notes (*History of the Theory of Probability*, § 858) that they do not profess originality except in working out the simple-product theory while they admit it to be arbitrary. Arbitrary I think it is; but in not adding a term containing $1-p_1$ it would be perfectly reasonable as an approximation. I believe the term would always be practically quite unimportant, unless in some instance quite unimportant itself.

5928. (By Professor TOWNSEND, F.R.S.)—A solid ellipsoid of uniform density being supposed to attract, according to the ordinary law of the inverse square of the distance, any element of its mass; show that the locus of points in its interior, for which the component of its attraction towards its centre is the same as for a concentric sphere of equal density, is the quadric cone coaxal with its surface whose equation referred to its principal planes is $(B+C-2A)\,x^2 + (C+A-2B)\,y^2 + (A+B-2C)\,z^2 = 0$, where A, B, C are the coefficients of the components of its attraction parallel to its axes.

Solution by the PROPOSER; W. J. C. SHARP, M.A.; *and others.*

For since, by hypothesis, if r be the distance of xyz from the centre,

$$Ax\,.\,\frac{x}{r} + By\,.\,\frac{y}{r} + Cz\,.\,\frac{z}{r} = \tfrac{4}{3}\pi\rho r,$$

and since, by a well-known relation, $A+B+C = 4\pi\rho$, therefore at once $Ax^2+By^2+Cz^2 = \frac{1}{3}(A+B+C)\,r^2 = \frac{1}{3}(A+B+C)(x^2+y^2+z^2)$, therefore, &c.

5698. (By CHRISTINE LADD.)—If A, B, C, D are four tangents to a conic, and if a, b, c, d are their respective points of contact, show that the triangle formed by the intersections of opposite connectors of the four centres of perspective of the four pairs of triangles abc, ABC; bcd, BCD; cda, CDA; dac, DAC, and the triangle formed by the connectors of oppo-

ır pairs of
.s of oppo-
rsections of

s the line through
is the point of inter-

. A; and DC, c, CB are
END's *Modern Geometry*,
three pairs of connectors,
and AB . CD are collinear.
he centre of perspective of
.s the centre of perspective of
.. shown that the line joining
. to that of dbc and DBC passes
and AB . CD, or, what is the
. That is to say, the intersection
the centres of perspective coincides
osite connectors of $abcd$. The same
. intersections of opposite connectors

) are two triads of lines through two
(C . ab) and (A . bc), (A . dc) and (C . da),
oint. But the first is the axis of perspec-
cond that of adc and ADC; and in the
t the axes of adb and ADB, dbc and DBC
. bc) (ab . cd), which is the same as the line

.TH BLACKWOOD.)—A point P is taken at random
. through this point a straight line PQ is drawn in
meeting the surface of the sphere in Q; show that
f PQ is three-fourths of the radius.

Solution by E. B. SEITZ.

.entre of the sphere, and let

$$OQ = r,\ OP = x, \text{ and } \angle OPQ = \theta.$$

. $(r^2 - x^2 \sin^2\theta)^{\frac{1}{2}} + x\cos\theta$; for given values of x and θ the num-
.ns of P is $4\pi x^2$, and for each position the number of directions
in θ. The limits of θ are 0 and π, and those of x are 0 and r.
. . value of $x\cos\theta$ between these limits is 0, the average length
.e same as that of $(r^2 - x^2\sin^2\theta)^{\frac{1}{2}}$, which can be found by taking
or the limits of θ. Hence the required average is

$$\Delta = \frac{\int_0^{\frac{1}{2}\pi}\int_0^r (r^2 - x^2\sin^2\theta)^{\frac{1}{2}}\, 2\pi\sin\theta\, d\theta \,.\, 4\pi x^2 dx}{\int_0^{\frac{1}{2}\pi}\int_0^r 2\pi\sin\theta\, d\theta\,.\, 4\pi x^2 dx}$$

$$= \frac{3}{r^3}\int_0^{\frac{1}{2}\pi}\int_0^r (r^2 - x^2\sin^2\theta)\ \sin\theta\, d\theta\, x^2\, dx$$

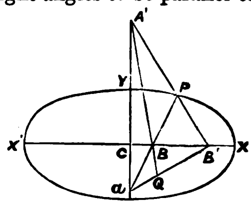
$$= \frac{3r}{8}\int_0^{\frac{1}{2}\pi}(\theta\operatorname{cosec}^2\theta - \cot\theta + \sin 2\theta)\, d\theta = \tfrac{3}{4}r.$$

6012. (By Prof. WOLSTENHOLME, M.A.)—Through a point P of an ellipse are drawn straight lines PBA, B′PA′ meeting the major and minor axes in B, B′; A, A′; such that PB = PB′ = b, PA = PA′ = a, and AB′, A′B meet in Q; prove that Q is the point on the normal at P through which pass all chords which subtend a right angle at P.

Solution by R. KNOWLES, L.C.P.; R. E. RILEY, B.A.; *and others.*

If we suppose the chords drawn from P at right angles to be parallel to the axes, and put $(x_1\, y_1)$ for the coordinates of P, we easily obtain

$$x = \left(\frac{a^2-b^2}{a^2+b^2}x_1\right) \text{ and } y = -\left(\frac{a^2-b^2}{a^2+b^2}\, y_1\right)$$



for the coordinates of the point on the normal through which pass all the chords that subtend a right angle at P. [This is an unsolved example in TODHUNTER's *Conics*, p. 271.] To show that Q is this point—we readily find the coordinates and equations as follows:—

$$\text{B} \ldots \left(\frac{a-b}{a}x_1, 0\right);\ \text{A}\left(0, \frac{b-a}{b}y_1\right);\ \text{B}'\left(\frac{a+b}{a}x_1, 0\right);\ \text{A}'\left(0, \frac{a+b}{b}y_1\right),$$

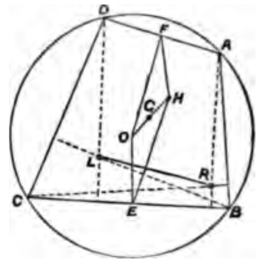
$$\text{AB}' \ldots \frac{ax}{(a+b)\,x_1} - \frac{by}{(a-b)\,y_1} = 1, \qquad \text{A}'\text{B} \ldots \frac{ax}{(a-b)\,x_1} + \frac{by}{(a+b)\,y_1} = 1;$$

whence the coordinates of Q are $x = \dfrac{a^2-b^2}{a^2+b^2}x_1$ and $y = -\dfrac{a^2-b^2}{a^2+b^2}y_1$.

5995. (By the Rev. F. D. THOMSON, M.A.)—A circle cuts a rectangular hyperbola in four points. Prove that the straight line joining their centres is bisected by the centroid of the four points.

Solution by R. GRAHAM, M.A.; C. SHARP, M.A.; *the* PROPOSER; *and others.*

Let ABCD be the quadrilateral; O the centre of the circle; E and F the middle points of the opposite sides BC and DA; L, M, N, R the orthocentres of the triangles BCD, CDA, DAB, ABC. Join LR. Then, since a circle passes through CLRB, it is easily seen that LR is equal and parallel to AD. Hence the quadrilateral LMNR has its sides equal and parallel to those of ABCD, therefore LA, MB, NC, RD meet in a point H, which bisects them all.



Now a rectangular hyperbola can be drawn through A, B, C, D, L, M, N, K, therefore H is its centre. Join HE, HF, OE, OF. Then, since H is the intersection of the diagonals of the parallelogram DLRA, HF is parallel to DL or OE; similarly HE is parallel to OF. Therefore, the diagonals of the parallelogram HFOE meet in G, the centroid of the four points A, B, C, D.

[With this may be compared the solution of Question 5737 (*Reprint*, Vol. XXX., p. 64), where it is shown that the same point H is the intersection of the four lines joining the feet of the perpendiculars of the triangles formed by taking three of the four points.]

6025. (By J. J. WALKER, M.A.)—Prove that the two determinants

$$\begin{vmatrix} 1 & 1 & a-b \\ x & ay & a(1-b)z \\ b(1-x) & 1-y & (a-1)bz \end{vmatrix} \text{ and } \begin{vmatrix} 2x-1 & 2y-1 & (a-b)(2z-1) \\ x & ay & a(1-b)z \\ b(x-1) & y-1 & (a-1)bz \end{vmatrix}$$

(or D_1, D_2, say) are identically equal in value. Can this equality be shown by reducing them to a common form?

Solution by H. C. ROBSON, B.A.; GABRIEL TORELLI; *and others.*

$$D_1 = \begin{vmatrix} 1 & 1 & a-b \\ x & ay & a(1-b)z \\ b(1-x) & 1-y & (1-a)bz \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 0 \\ x & ay & 0 \\ 0 & 0 & (1-a)bz \end{vmatrix}$$

$$= -2bz(1-a)(ay-x);$$

$$D_2 = \begin{vmatrix} 1 & 1 & a-b \\ x & ay & a(1-b)z \\ b(1-x) & 1-y & (1-a)bz \end{vmatrix} - 2\begin{vmatrix} x & y & (a-b)z \\ x & ay & a(1-b)z \\ b(1-x) & 1-y & (1-a)bz \end{vmatrix}$$

In the last determinant from the last column subtract $\frac{az}{x}$ times the first, add to it $\frac{bz}{y}$ times the second, and we get

$$-\begin{vmatrix} x & y & 0 \\ x & ay & 0 \\ b(1-x) & 1-y & -\frac{abz}{x}+\frac{bz}{y} \end{vmatrix} = -2(1-a)b(ay-x)z.$$

6041. (By Prof. ASAPH HALL, M.A.)—Find the moments of inertia of an elliptic disc about a straight line in the plane of the disc and parallel to (1) the axis of x, (2) the axis of y; the equation of the disc being

$$ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

Solution by R. KNOWLES, L.C.P.; T. R. TERRY, M.A.; *and others.*

The moment of inertia of ellipse about any diameter CP is known to be $\frac{\rho}{4\pi^2}$ (area)^3 $\frac{1}{\mathrm{CP}^2}$; and, as referred to the centre, the equation becomes

$$ax^2+2bxy+cy^2+\frac{\Delta}{ac-b^2}=0,$$

where $\Delta=acf-2bde-ae^2-b^2f-cd^2$; therefore area $=\frac{-\pi\Delta}{(ac-b^2)^{\frac{3}{2}}}$.

For a line parallel to the axis of x, we have

$$\frac{1}{\mathrm{CP}^2}=\frac{a\,(ac-b^2)}{-\Delta};\ \text{and for axis of } y,\ =\frac{b\,(ac-b^2)}{-\Delta};$$

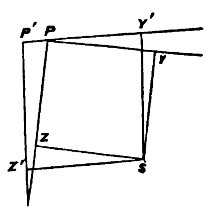
hence the moments of inertia for lines through the centre parallel to the axes are $\frac{\pi\rho a\Delta^2}{(ac-b^2)^{\frac{7}{2}}}$ and $\frac{\pi\rho b\Delta^2}{(ac-b^2)^{\frac{7}{2}}}$.

If we want the moments round lines at distances p and q from these, we must add $\frac{-\pi\rho\Delta}{(ac-b^2)^{\frac{3}{2}}}p^2$ and $\frac{-\pi\rho\Delta}{(ac-b^2)^{\frac{3}{2}}}q^2$ respectively.

5929. (By Professor WOLSTENHOLME, M.A.)—Prove that the area of the pedal of a closed oval from a point within it exceeds the area of the pedal of its evolute from the same point by the area of the oval.

I. *Solution by* E. B. ELLIOTT, M.A.; A. W. CAVE, B.A.; *and others.*

Let P, P′ be two consecutive points on the curve; S the pedal origin; Y, Y′ the feet of perpendiculars from it on the tangents at P, P′; and Z, Z′ the feet of those on the normals. Then SZ is parallel and equal to PY, and SZ′ is parallel and equal to P′Y′, *i.e.*, in the limit to PY. Therefore the triangles ZSZ′, YPY′ are equal. But ZSZ′ is an element of the area of the pedal of the evolute of P, and YPY′ is an element of the area between the locus of P and its pedal.



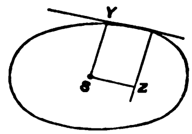
Hence, summing, we have

(1) If the locus of P be a curve, the area between it and its pedal is equal to the area of the pedal of its evolute; or, in other words, the area

of the pedal exceeds that of the pedal of the evolute by the area of the curve, as was to be shown; and more generally,—

(2) If the locus of P be not closed, the area contained between it, its pedal, and its two extreme tangents, is equal to that contained between the pedal of its evolute and the two radii vectores from the pedal origin to the ends of that pedal.

II. *Solution by the* PROPOSER.



$$\text{Area of Pedal (1)} = \int_0^{2\pi} p^2 d\phi,$$

$$\text{Area of Pedal (2)} = \int_0^{2\pi} \frac{dp^2}{d\phi^2}\, d\phi$$

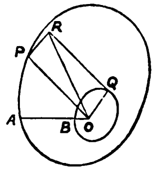
$$= \left(p\frac{dp}{d\phi}\right)_0^{2\pi} - \int_0^{2\pi} p\frac{d^2p}{d\phi^2}\, d\phi = -\int_0^{2\pi} p\frac{d^2p}{d\phi}\, d\phi.$$

Hence difference of areas

$$= \int_0^{2\pi} p\left(p + \frac{d^2p}{d\phi^2}\right) d\phi \quad \int_0^{2\pi} p\frac{ds}{d\phi}\,.\,d\phi$$

$$= \int pds = \text{area of oval.}$$

III. *Solution by* ROBERT RAWSON.



From any point O in the interior of two closed ovals AP, BQ, draw OP, OQ at right angles, and construct the parallelogram OPRQ. Consider first the oval traced by R.

Put $\quad OP = p;\ \ OQ = p_1;\ \ OR = r;$

$\angle AOP = \phi;\ \ \angle AOQ = \phi + \frac{1}{2}\pi;\ \ \angle AOR = \theta.$

It will be readily seen that

$$\theta = \phi + \tan^{-1}\frac{p_1}{p},\quad r^2 = p^2 + p_1^2 \text{ .... } (1, 2).$$

Put $$A = \int_a^{2\pi+a} r^2 d\theta = \text{twice the area of oval traced by R},$$

$$A_1 = \int_a^{2\pi+a} p^2 d\phi = \text{twice the area of oval AP},$$

$$A_2 = \int_a^{2\pi+a} p_1^2 d\phi = \text{twice the area of oval BQ}.$$

From (2) we have

$$A = \int_a^{2\pi+a} (p^2 + p_1^2)\, d\theta = \int_0^{2\pi} (p^2 + p_1^2) \left\{ d\phi + \frac{p\, d\left(\frac{p_1}{p}\right)}{p^2 + p_1^2} \right\};$$

therefore $$A = A_1 + A_2 + \int_0^{2\pi} p^2 \frac{d}{d\phi}\left(\frac{p_1}{p}\right) d\phi \quad\text{.....................(3).}$$

Integrating by parts, we have $A = A_1 + A_2 + \left[pp_1\right]_0^{2\pi} - 2\int_0^{2\pi} p_1 \frac{dp}{d\phi} d\phi$.

Now, as pp_1 is the same at both limits, we have

$$A = A_1 + A_2 - 2\int_0^{2\pi} p_1 \frac{dp}{d\phi}\, d\phi \quad\text{.........................(4).}$$

Equation (4) obtains without any restriction whatever being put upon the oval curves AP, BQ. If, however, the oval AP is the pedal to R, then the oval BQ will be the pedal to the evolute R. This being the case, then

$\frac{dr}{d\theta} = \frac{rp_1}{p}$. (See TODHUNTER's *Differential Calculus* p. 299) ......(5).

Eliminating θ between (1), (5), $\frac{dp}{d\phi} = p_1$ ....(6).

Substituting this value in (4), we have $A = A_1 - A_2$ ....(7), which is the neat and important property enunciated in the question.

In equation (3) put $p^2 \frac{d}{d\phi}\left(\frac{p_1}{p}\right) = -2p_1^2$;

then, integrating, $p = (2\phi + c)\, p_1$ ....(8).

Hence, the property in the question will obtain for any ovals subject to the relation (8). Interesting properties may be obtained from (3) and (4).

5986. (By Prof. WOLSTENHOLME, M.A.)—Prove that, when n is a positive integer, α an angle $< \frac{1}{2}\pi$, and $1 + x\sin\alpha$ positive,

$$\int_0^{\alpha} \frac{\sin^{n-2}\theta\,(n-1-n\sin^2\theta - x\sin\theta)\, d\theta}{(1+x\sin\theta)^{n+1}} = \frac{\cos\alpha\,\sin^{n-1}\alpha}{(1+x\sin\alpha)^n}.$$

I. *Solution by* T. R. TERRY, M.A.; H. STABENOW, M.A.; *and others.*

Let $u = \frac{\cos\theta\,\sin^{n-1}\theta}{(1+x\sin\theta)^n}$, then $\frac{du}{d\theta} = \frac{\sin^{n-2}\theta\,(n-1-n\sin^2\theta - x\sin\theta)}{(1+x\sin\theta)^{n+1}}$.

Under the given conditions $\frac{du}{d\theta}$ never becomes infinite between 0 and α, and u vanishes when $\theta = 0$, hence we have

$$\int_0^{\alpha} \frac{du}{d\theta} = \frac{\cos\alpha\,\sin^{n-1}\alpha}{(1+x\sin\alpha)^n}.$$

II. *Solution by the* PROPOSER.

(1). Expand $(1+x\sin\theta)^{-(n+1)}$; then the coefficient of x^{r-1}, under the sign of integration, will be

$$\frac{(-1)^{r-1}}{n!}\{r(r+1)\dots(r+n-1)\sin^{n+r-3}\theta\,(n-1-n\sin^2\theta)$$
$$+(r-1)r(r+1)\dots(r+n-2)\sin^{n+r-3}\theta\},$$

or $$(-1)^{r-1}\frac{r(r+1)\dots(r+n-2)}{n!}$$
$$\times\{n(r+n-2)\sin^{n+r-3}\theta-n(n+r-1)\sin^{n+r-1}\theta\},$$

or $$(-1)^{r-1}\frac{(n+r-2)!}{(n-1)!\,(r-1)!}\{(n+r-2)\sin^{n+r-3}\theta-(n+r-1)\sin^{n+r-1}\theta\}.$$

But

$$(n+r-1)\int_0^a\sin^{n+r-1}\theta\,d\theta=-\sin^{n+r-2}a\cos a+(n+r-2)\int_0^a\sin^{n+r-3}\theta\,d\theta;$$

whence the coefficient of x^{r-1}, after integration, is

$$(-1)^{r-1}\frac{(n+r-2)!}{(n-1)!\,(r-1)!}\cos a\sin^{n+r-2}a,$$

which is the coefficient of x^{r-1} in the expansion of $\dfrac{\cos a\sin^{n-1}a}{(1+x\sin a)^n}$.

There is one exception, when $n=1$ and $r=1$; since then for $-\sin^{n+r-2}a\cos a$, we get $1-\cos a$, so that the value of the integral, when $n=1$, is $\dfrac{\cos a}{1+x\sin a}-1$.

(2). Otherwise: denote the integral by u_n; then we have

$$\frac{du_n}{dx}=-\int_0^a\frac{\sin^{n-1}\theta\,(1+x\sin\theta)+(n+1)\sin^{n-1}\theta\,(n-1-n\sin^2\theta-x\sin\theta)}{(1+x\sin\theta)^{n+2}}d\theta$$

$$\equiv-n\int_0^a\sin^{n-1}\theta\,\frac{\{n-(n+1)\sin^2\theta-x\sin\theta\}}{(1+x\sin\theta)^{n+2}}\,d\theta=-n\,u_{n+1}.$$

Thus $\quad u_n=(-1)^{n-1}\dfrac{d^{n-1}}{dx^{n-1}}u_1,\;$ and $\;u_1=-\displaystyle\int_0^a\frac{x+\sin\theta}{(1+x\sin\theta)^2}d\theta.$

Now let

$$y=\int_0^a\frac{d\theta}{1+x\sin\theta}=\frac{2}{(1-x^2)^{\frac12}}\left\{\tan^{-1}\frac{x+\tan\frac12a}{(1-x^2)^{\frac12}}-\tan^{-1}\frac{x}{(1-x^2)^{\frac12}}\right\};$$

then $$\frac{dx}{dy}=\frac{x}{1-x^2}y+\frac{2}{(1-x^2)^{\frac12}}$$
$$\times\left\{\frac{1-x^2+x^2+x\tan\frac12a}{(1-x^2)^{\frac12}\{1-x^2+(x+\tan\frac12a)^2\}}-\frac{1-x^2+x^2}{(1-x^2)^{\frac12}(1-x^2+x^2)}\right\};$$

and $$(1-x^2)\frac{dy}{dx}-xy=2\frac{(1+x\tan\frac12a)}{1+\tan^2\frac12a+2x\tan\frac12a}-2=\frac{1+\cos a+x\sin a}{1+x\sin a}-2,$$

$$\text{or}\quad (1-x^2)\int_0^{a}\frac{-\sin\theta\,d\theta}{(1+x\sin\theta)^2}-x\int\frac{d\theta}{1+x\sin\theta}$$

$$=\frac{\cos a}{1+x\sin a}-1=-\int_0^{a}\frac{x+\sin\theta}{(1+x\sin\theta)^2}\,d\theta;$$

$$\text{giving}\quad u_1=\frac{\cos a}{1+x\sin a}-1;\ \text{whence}\ u_n=\frac{\cos a\,(\sin a)^{n-1}}{(1+x\sin a)^n}.$$

Both these methods assume that x is numerically less than 1, but it is obvious that this is not a necessary limitation, each integral being a definite function of x, and finite provided that $1+x\sin a$ does not become 0 between 0 and a. The sufficient condition for this is that $1+x\sin a$ be positive, if a denote an angle less than $\frac{1}{2}\pi$. If $1+x\sin\beta=0$, where β is an angle between 0 and a, it is obvious that u_1 is infinitely great, and most probably all the others are then infinite also. Of course the result is obvious on differentiating each member with respect to a.

5752. (By G. S. Carr.)—Give a general method for the synthesis of a determinant, and exemplify it by throwing into the form of a determinant

$$abcd+bfgl+f^2h^2+edf+cghp+ahr+elpr$$
$$-fhpr-ablr-ach^2-fgh-bdf^2-efhl-cedp.$$

Solution by the Proposer.

The process is facilitated by making use of two evident rules. Those constituents which belong to the row and column of a given constituent a, will be designated a's constituents. Also two pairs of constituents such as a_p, c_q, and a_q, c_p, (the suffix denoting the column) forming the corners of a rectangle, will be said to be conjugate to each other.

Rule I.—No constituent will be found in the same term with one of its own constituents.

Rule II.—The conjugates of any two constituents a and b will be common to a's and b's constituents.

Ex.—To write in the form of a determinant the following terms:

$$abcd+bfgl+f^2h^2+1edf+cghp+1ahr+elpr$$
$$-fhpr-ablr-ach^2-1fhg-bdf^2-efhl-cedp.$$

The determinant will be of the fourth order; and since every term must contain four constituents, the constituent 1 is supplied to make up the number in some of the terms. Select any term, as $abcd$, for the leading diagonal.

$$\text{Now apply Rule I.}\ \left\{\begin{array}{ll} a \text{ is not found with } e,\ f,\ f,\ g,\ p,\ 0 \ \ldots\ldots\ldots\ldots\ldots & (1),\\ b \text{ is not found with } e,\ h,\ h,\ p,\ 1,\ 0 \ \ldots\ldots\ldots\ldots\ldots & (2),\\ c \text{ is not found with } f,\ f,\ l,\ r,\ 1,\ 0 \ \ldots\ldots\ldots\ldots\ldots & (3),\\ d \text{ is not found with } g,\ h,\ h,\ l,\ r,\ 0 \ \ldots\ldots\ldots\ldots\ldots & (4).\end{array}\right.$$

Each constituent has $2(n-1)$, that is 6, constituents belonging to it since $n=4$. Assuming, therefore, that the above letters are the constituents of a, b, c, d, and that there are no more, we supply a sixth zero constituent in each case.

Now apply Rule II.—The constituents common

to a and b are e, p; to a and d are g, 0; to a and c are f, f:
to b and d are h, h, 0; to b and c are 1, 0; to c and d are l, r, 0.

The determinant may now be formed. The diagonal being $abcd$; place e, p, the conjugates of a and b, either as in the diagram or transposed.

$$\begin{vmatrix} a & e & f & g \\ p & b & 1 & h \\ f & 0 & c & r \\ 0 & h & l & d \end{vmatrix}$$

Then f and f, the conjugates of a and c, may be written. 1 and 0, the conjugates of b and c, must be placed as indicated, because 1 is one of p's constituents, since it is not found in any term with p, and must therefore be in the second row.

Similarly, the places of g and 0, and of l and r, are assigned.

In the case of b and d, we have h, h, 0 from which to choose the two conjugates, but we see that 0 is not one of them because that would assign two zero constituents to b, whereas b has but one, which is already placed.

By similar reasoning, the ambiguity in selecting the conjugates l, r is removed.

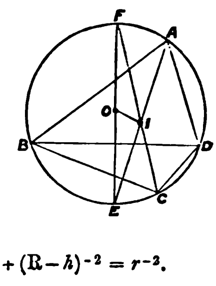
The foregoing method is rigid in the case of a complete determinant having different constituents. It becomes uncertain when the zero constituents increase in number, and when several constituents are identical. But even then, in the majority of cases, it will soon afford a clue to the required arrangement.

5137. (By Professor SCHEFFER.)—If r and R be the radii of the inscribed and circumscribed circles of a quadrilateral, and h the distance between their centres; prove that $(R+h)^{-2}+(R-h)^{-2} = r^{-2}$.

6020. (By C. LEUDESDORF, M.A.)—A quadrilateral is circumscribed to a circle whose radius is r, and its vertices lie on another circle whose radius is R; if D $(=h)$ be the distance between the centres of the circles, show that the rectangle contained by the diagonals of the quadrilateral is

$$8R^2r^2\,(R^2-h^2)^{-1}.$$

I. Solution of both Questions by R. F. DAVIS, M.A.



Let ABCD be a quadrilateral inscribed in a circle of centre O; EOF the diameter at right angles to BD. Then, if a circle can be inscribed in the quadrilateral, I, the point of intersection of AE, CF, will be its centre. Let R, r denote the two radii, and h the distance OI.

(Quest. 5137). Then $AI = r\operatorname{cosec}\frac{1}{2}A$, and $CI = r\operatorname{cosec}\frac{1}{2}C = r\sec\frac{1}{2}A$. Therefore

$(AI)^{-2}+(CI)^{-2} = r^{-2}$, and $AI \cdot CI = 2r^2\operatorname{cosec}A$.

Now $\quad AI \cdot EI = CI \cdot FI = R^2-h^2$

so that $\quad EI^2+FI^2 = r^{-2}\,(R^2-h^2)^2$.

But $\quad EI^2+FI^2 = 2R^2+2h^2$; therefore $(R+h)^{-2}+(R-h)^{-2} = r^{-2}$.

(Quest. 6020). Since $AC = 2R\sin ABC = 2R\sin AEC = 2R\frac{CI}{EI}$, and $BD = 2R\sin A = \frac{4Rr^2}{AI\,.\,CI}$, therefore $AC\,.\,BD = \frac{8R^2r^2}{AI\,.\,EI} = \frac{8R^2r^2}{R^2-k^2}$.

II. *Solution of Quest.* 6020 *by the* Rev. F. D. THOMSON, M.A.

Taking rectangular axes through the centre of the inner circle, the equation to the outer circle is

$$(x\cos\alpha + y\sin\alpha - r)(x\cos\gamma + y\sin\gamma - r)$$
$$+ (x\cos\beta + y\sin\beta - r)(x\cos\delta + y\sin\delta - r) = 0,$$

with the condition $\delta - \gamma + \beta - \alpha = \pi$.

The coefficient of x^2 in (1) = coefficient of y^2 = M suppose;

therefore
$$\begin{aligned} M &= \cos\alpha\cos\gamma + \cos\beta\cos\delta = \sin\alpha\sin\gamma + \sin\beta\sin\delta \\ &= \tfrac{1}{2}[\cos(\alpha-\gamma) + \cos(\beta-\delta)], \quad \text{since } \alpha+\gamma = \pi-(\beta+\delta) \\ &= \tfrac{1}{2}\{\cos[(\alpha-\beta)+(\beta-\gamma)] + \cos[(\beta-\gamma)+(\gamma-\delta)]\} \\ &= -\sin(\alpha-\beta)\sin(\beta-\gamma), \end{aligned}$$

since $\sin(\alpha-\beta) = \sin(\gamma-\delta)$, $\cos(\alpha-\beta) = -\cos(\gamma-\delta)$;

therefore constant term $\div M = \frac{2r^2}{M} = d^2 - R^2 = -\frac{2r^2}{\sin(\alpha-\beta)(\sin\beta-\gamma)}$.

But rectangle of diagonals $= 4R^2\sin(\beta-\alpha)\sin(\gamma-\beta)$

$$= 4R^2 \times \frac{2r^2}{R^2-d^2} = \frac{8R^2r^2}{R^2-d^2}.$$

We may prove either as above, or geometrically by taking a particular case that $2r^2(R^2+d^2) = (R^2-d^2)^2$.

[The corresponding condition for any two conics is

$$\Theta'^3 = 4\Delta'(\Theta\Theta' - 2\Delta\Delta'),$$

where the accented letters refer to the inner conic, as proved in the Solution of Quest. 2840, on p. 25 of Vol. XII. of the *Reprint.*]

6062. (By R. E. RILEY, B.A.)—CP, CD are conjugate radii of an ellipse; E, F are the middle points of PN, DM, perpendiculars on the major axis. Show that the straight line EF touches at its middle point the ellipse $2b^2x^2 + 8a^2y^2 = a^2b^2$.

I. *Solution by* G. G. STORR; D. EDWARDES; *and others.*

If ϕ be the eccentric angle of P, the equation of EF is

$$\frac{y-\frac{1}{2}b\sin\phi}{x-a\cos\phi} = -\frac{\frac{1}{2}b(\cos\phi-\sin\phi)}{a(\sin\phi+\cos\phi)}, \text{ or } (bx+2ay)\cos\phi + (2ay-bx)\sin\phi = ab,$$

and the envelop of this line is

$$(bx+2ay)^2+(2ay-bx)^2 = a^2b^2 \quad \text{or} \quad 2b^2x^2+8a^2y^2 = a^2b^2.$$

The coordinates of the middle point of EF are

$$\tfrac{1}{2}a\,(a\phi-\sin\phi),\ \tfrac{1}{4}b\,(\cos\phi+\sin\phi)\text{ ; and since}$$

$$2b^2.\tfrac{1}{4}a^2(\cos\phi-\sin\phi)^2+8a^2.\tfrac{1}{16}b^2(\cos\phi+\sin\phi)^2 = a^2b^2, \text{ therefore \&c.}$$

II. *Solution by* FRANCES E. PRUDDEN ; T. R. TERRY, M.A. ; *and others.*

If ϕ be the eccentric angle of P, the coordinates of E, F are

$$(a\cos\phi,\ \tfrac{1}{2}b\sin\phi),\quad (-a\sin\phi,\ \tfrac{1}{2}b\cos\phi)\ ;$$

and the coordinates of the middle point of EF are

$$\frac{a}{\sqrt{2}}\cos(\phi+\tfrac{1}{4}\pi),\quad \frac{b}{2\sqrt{2}}\sin(\phi+\tfrac{1}{4}\pi).$$

The equation to EF is $\dfrac{x\sqrt{2}}{a}\cos(\phi+\tfrac{1}{4}\pi)+\dfrac{y\,2\sqrt{2}}{b}\sin(\phi+\tfrac{1}{4}\pi) = 1.$

But we know that $\dfrac{x}{a'}\cos\theta+\dfrac{y}{b'}\sin\theta = 1$ is the tangent to the ellipse $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2} = 1$ at the point $(a'\cos\theta,\ b'\sin\theta)$.

Therefore EF touches the ellipse $2b^2x^2+8a^2y^2 = a^2b^2$ at the point

$$\frac{a}{\sqrt{2}}\cos(\phi+\tfrac{1}{4}\pi),\quad \frac{b}{2\sqrt{2}}\sin(\phi+\tfrac{1}{4}\pi)\ ;$$

that is to say, at the middle point of EF.

5816. (By E. B. SEITZ.)—Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is $\frac{13}{35}$, and (2) the average distance between them is $\frac{17}{8}r$.

Solution by the PROPOSER.

1. Let A and B be the centres of the two spheres. With B as a centre and a radius x describe a sphere; the area of the zone of this sphere within the sphere A is

$$u = \frac{\pi}{2r}\,(4rx^2-3r^2x-x^3).$$

From any point of this zone as a centre with a radius $2r$ describe a sphere; the volume common to this sphere and the sphere B is

$$v = \tfrac{1}{12}\pi\left(x^3-30r^2x+72r^3-\frac{27r^4}{x}\right).$$

Now, if one of the points be taken in the zone u, and the other within the

volume v, the distance between them will be less than $2r$. The limits of x are r and $3r$; hence the required chance is

$$p = \frac{\int_r^{3r} u\,v\,dx}{\int_r^{3r} \frac{4}{3}\pi r^3 u\,dx} = \frac{3}{128r^7}\int_r^{3r} (81r^6 - 324r^5x + 405r^4x^2 - 192r^3x^3 + 27r^2x^4 + 4rx^5 - x^6)\,dx = \tfrac{13}{35}.$$

2. From any point of the zone u as a centre with a radius y describe a sphere intersecting the sphere B; the area of the zone of this sphere within the sphere B is $\quad w = \frac{\pi}{x}(2xy^2 - y^3 - x^2y + r^2y).$

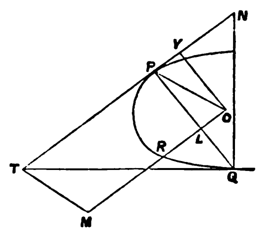
The limits of y are $x-r$ and $x+r$; hence the required average is

$$\Delta = \frac{\int_r^{3r}\int_{x-r}^{x+r} u\,w\,y\,dx\,dy}{\int_r^{3r}\int_{x-r}^{x+r} u\,w\,dx\,dy} = \frac{9}{32r^7}\int_r^{3r}\int_{x-r}^{x+r} (4rx - 3r^2 - x^2)(2xy - y^2 - x^2 + r^2)\,y^2\,dx\,dy$$

$$= \frac{3}{40r^4}\int_r^{3r} (20rx^3 - 16r^2x^2 + 4r^3x - 5x^4 - 3r^4)\,dx = \tfrac{11}{8}r.$$

5946. (By J. J. WALKER, M.A.)—If PQ is a chord of a central conic, normal at P, and if the normal at Q meets the tangent at P in N, while Y is the foot of the perpendicular on the tangent from its centre; prove that PN : PY = PQ² : OR², OR being the semi-diameter parallel to the tangent at P.

Solution by E. ANTHONY, M.A.; LIZZIE A. KITTUDGE; *and others.*



Draw TM parallel to OP; then, since OP, OR are conjugate, the polar of L is parallel to OP; and the polar of L passes through T; therefore TM is the polar of L, therefore

$$OL \,.\, OM = OR^2,$$

that is, $\quad PY \,.\, PT = OR^2;$

and $\quad PN \,.\, PT = PQ^2,$

therefore $\quad PN : PY = PQ^2 : OR^2.$

5964. (By J. J. WALKER, M.A.)—Prove that in any plane curve the intercept on the line of reference between the pole and normal is equal to

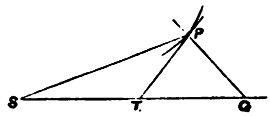
$$r\,dr \div d\,(r\cos\theta).$$

Solution by the Rev. J. L. Kitchin, M.A.; J. Hammond, M.A; *and others.*

$$\frac{SG}{SP} = \frac{\sin SPG}{\sin SGP} = \frac{\cos\phi}{\cos(\phi+\theta)},$$

therefore $SG = \dfrac{r\cos\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$

$$= \frac{r}{\cos\theta - \sin\theta\,\dfrac{rd\theta}{dr}} = \frac{rdr}{d(r\cos\theta)}.$$



5779. (By R. Rawson.) — Show that $C_1P_n + C_2P_n \int \frac{(a-\beta^2)^{m-\frac{1}{2}}}{(P_n)^2}\,d\beta$ is the complete primitive of the differential equation

$$\frac{d^2P_n}{dx^2} + \left\{ \frac{(2m-1)\,\beta\,\dfrac{d\beta}{dx}}{a-\beta^2} - \frac{\dfrac{d^2\beta}{dx^2}}{\dfrac{d\beta}{dx}} \right\} \frac{dP_n}{dx} + \frac{n(n-2m)\left(\dfrac{d\beta}{dx}\right)^2}{a-\beta^2}\,.\,P_n = 0,$$

where P_n is the coefficient of y^n in $(a+2\beta y+y^2)^m$, β is a function of x, and a, m any constants whatever.

Solution by the Proposer.

Put $$\Sigma P_n y^n = (a+2\beta y+y^2)^m \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiate (1) with respect to x and y as follows:—

$$x_1\Sigma\frac{d^2P_n}{dx^2}y^n = m(a+2\beta y+y^2)^{m-2}$$
$$\times x_1\left\{2a\frac{d^2\beta}{dx^2}y + \left[4(m-1)\left(\frac{d\beta}{dx}\right)^2 + 4\beta\frac{d^2\beta}{dx^2}\right]y^2 + 2\frac{d^2\beta}{dx^2}y^3\right\},$$

$$x_2\Sigma\frac{dP_n}{dx}y^n = m(a+2\beta y+y^2)^{m-2}\times x_2\left\{2a\frac{d\beta}{dx}y + 4\beta\frac{d\beta}{dx}y^2 + 2\frac{d\beta}{dx}y^3\right\},$$

$$x_3\Sigma nP_ny^n = m(a+2\beta y+y^2)^{m-2}\times x_3\left\{2a\beta y + (4\beta^2+2a)y^2 + 6\beta y^3 + 2y^4\right\},$$

$$\Sigma n(n-1)P_ny^n = m(a+2\beta y+y^2)^{m-2}$$
$$\times\left\{\left[4(m-1)\beta^2+2a\right]y^2 + 4(2m-1)\beta y^3 + (4m-2)y^4\right\},$$

where x_1, x_2, x_3 are multipliers to be determined.

By adding the above four equations together, there results

$$x_1\frac{d^2P_n}{dx^2} + x_2\frac{dP_n}{dx} + \left\{nx_3 + n(n-1)\right\}P_n = 0 \;\ldots\ldots\ldots\ldots\ldots (2).$$

If x_1, x_2, x_3 are determined so as to satisfy the following equations, viz.,

$$\left\{2(m-1)\left(\frac{d\beta}{dx}\right)^2 + 2\beta\frac{d^2\beta}{dx^2}\right\}x_1 + 2\beta\frac{d\beta}{dx}\,.\,x_2 + (2\beta^2+a)x_3 + a + 2(m-1)\beta^2 = 0 \;\ldots\ldots\ldots (3),$$

$$\frac{d^2\beta}{dx^2}\,.\,x_1 + \frac{d\beta}{dx}\,.\,x_2 + \beta x_3 = 0, \quad x_3 + 2m - 1 = 0 \ldots\ldots\ldots\ldots (4, 5),$$

we have $x_1 = \dfrac{a-\beta^2}{\left(\dfrac{d\beta}{dx}\right)^2}$, $x_2 = \dfrac{(2m-1)\beta}{\dfrac{d\beta}{dx}} - \dfrac{(a-\beta^2)\dfrac{d^2\beta}{dx^2}}{\left(\dfrac{d\beta}{dx}\right)^3}$, $x_3 = 1-2m$.

Substituting these values in (2), it becomes

$$\frac{d^2P_n}{dx^2} + \left\{\frac{(2m-1)\beta\dfrac{d\beta}{dx}}{a-\beta^2} - \frac{\dfrac{d^2\beta}{dx^2}}{\dfrac{d\beta}{dx}}\right\}\frac{dP_n}{dx} + \frac{n(n-2m)\left(\dfrac{d\beta}{dx}\right)^2}{a-\beta^2}.P_n = 0\ldots(6),$$

P_n, therefore, is a particular solution of (6).

It is shown in Boole's *Differential Equations*, p. 205, that another particular solution of this equation is

$$P_n\int\frac{e^{-\int\left(\frac{(2m-1)\beta\frac{d\beta}{dx}}{a-\beta^2} - \frac{\frac{d^2\beta}{dx^2}}{\frac{d\beta}{dx}}\right)dx}}{(P_n)^2}.\,dx = P_n\int\frac{(a-\beta^2)^{\frac{1}{2}(2m-1)}}{(P_n)^2}\,d\beta.$$

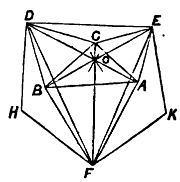
Hence $C_1P_n + C_2P_n\displaystyle\int\frac{(a-\beta^2)^{\frac{1}{2}(2m-1)}}{(P_n)^2}\,d\beta$ is the complete solution of equation (6).

If $m = -\frac{1}{2}$, $a = 1$, $\beta = -x$, then (6) becomes the differential equation which is satisfied by Legendre's nth coefficient.

5811. (By Hugh Murphy.)—Given the three lines that join the vertices of the equilateral triangles described on the three sides of any triangle; construct the original triangle.

Solution by J. O'Regan; J. J. Sides; *and others.*

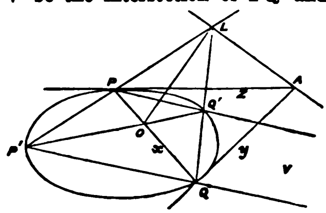
Let DEF be the triangle formed by the three given lines. It is known that the three lines joining the vertices of the equilateral triangles to the opposite vertices of the original triangle are equal, that they meet in one point, and intersect at angles of 120°. Hence, to determine their common point, construct the isosceles triangles DHF and EKF, making the angles DHF and EKF each equal to 120°; with H and K as centres and radii equal to HD and KE describe arcs intersecting at O; then ∠ DOF = EOF = EOD = 120°; therefore O is the common point of the three lines above named. Again, it is known that the sum of the distances from the intersection of these three lines to the vertices of the original triangle is equal to one of the lines; hence the sum of the lines DO, EO, FO is equal to twice one of the lines. Therefore produce DO, EO, FO, making DA = EB = FC = $\frac{1}{2}$ (DO + EO + FO); join AE, AF, BF, BD, CE; then is ABC the required triangle.



5416. (By H. W. Harris, B.A.)—If P and Q are two fixed points on one conic, and L a movable point on another, and PL and QL intersect the first conic in P′ and Q′, find the conics of intersection of QP′ and PQ′.

Solution by J. W. Sharpe, B.A.; C. Harkema; *and others.*

Join PQ, P′Q′, meeting in O. Let V be the intersection of PQ′ and P′Q. Draw AP, AQ tangents at P, Q. Take APQ for triangle of reference, so that the equations to PQ, QA, AP are respectively



$$x = 0, \quad y = 0, \quad z = 0.$$

Then the equation to the conic to which P, Q belong will be of the form $\quad x^2 + 2kyz = 0.$

Let L be the point (α, β, γ). Then the equations to LP, LQ, LA are

$$\frac{z}{\gamma} - \frac{x}{\alpha} = 0, \quad \frac{x}{\alpha} - \frac{y}{\beta} = 0, \quad \frac{y}{\beta} - \frac{z}{\gamma} = 0.$$

Now O is the pole of LA; therefore the equations to O are $x = 0$, $\frac{y}{\beta} = \frac{z}{-\gamma}$; hence $\frac{y}{\beta} + \frac{z}{\gamma} - \frac{2x}{\alpha} = 0$ is the equation to LO. But V is the pole of LO; therefore the equations to V are

$$\frac{\alpha}{-\frac{2}{x}} = \frac{\beta}{\frac{1}{hz}} = \frac{\gamma}{\frac{1}{hy}}.$$

Now let α, β, γ satisfy the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

hence x, y, z, the coordinates of V, satisfy

$$x^2(by^2 + cz^2 + 2fyz) + 4kyz(kayz - gzx - hxy) = 0.$$

This is a quartic curve, which touches AP, AQ at P, Q, and also at A; and cuts PQ also in the points where PQ cuts the conic upon which L moves.

If the two conics have double contact at P and Q, then the locus of V reduces to a conic having double contact with the other two at P and Q.

If the conic upon which L moves circumscribes the triangle APQ, then the locus of V reduces to a straight line.

5780. (By the Rev. F. D. Thomson, M.A.)—From the general equation to a curve of any order, the equation is formed for determining the lengths of the normals drawn from a given point to the curve. Will one of the roots of this equation become infinite when the condition is introduced that the curve touches the line at infinity? If so, why?

Solution by the PROPOSER.

Let A, B be two fixed points upon the line $z = 0$, and let CA, CB be the axes of x and y. Then the equation to a curve touching AB is of the form

$$S \equiv (ax + by)^2 u + zv = 0 \quad \text{.........................(1)}.$$

Let the tangent at P meet AB in T, and let N be the fourth harmonic to A, T, B. Then PN may be called the *quasi-normal* at P.

Now T is given by $Z = 0$, $X\frac{dS}{dx} + Y\frac{dS}{dy} = 0$, where x, y, z are the coordinates of P. Therefore N is given by

$$Z = 0, \quad X\frac{dS}{dx} - Y\frac{dS}{dy} = 0,$$

and the equation to PN is

$$z\left(X\frac{dS}{dx} - Y\frac{dS}{dy}\right) = Z\left(x\frac{dS}{dx} - y\frac{dS}{dy}\right) \text{.................(2)};$$

or, regarding x, y, z as current coordinates, (2) is the equation to a curve cutting (1) in the points to which quasi-normals can be drawn from the point (X, Y, Z.)

Now, at the point of contact of AB with S, $\frac{dS}{dx} = \frac{dS}{dy} = 0$;

therefore (2) is identically satisfied; or one of the points of intersection is the point of contact of S with AB.

Hence, when A and B are the circular points, and PN becomes a normal, the line joining *any point* to the point of contact at infinity satisfies the condition of being a normal, or one of the normals becomes infinite.

I do not see, however, how this is to be reconciled with the statement (SALMON's *Higher Plane Curves*, p. 110) that "*when a point in the curve is at infinity the normal at it will lie altogether at infinity*," and for that reason I have proposed this question.

In the same manner, it may be shown that, if S pass through the circular points, any line to either of them satisfies the condition of being a normal, and the number of finite normals is reduced by two.

It appears, therefore, that *any straight line* may be regarded as perpendicular to the line at infinity. We have, in fact, for the condition that $ax + by + cz = 0$ and also $a'x + b'y + c'z = 0$ may be perpendicular, the equation

$$aa' + bb' + cc' - (bc' + b'c)\cos A - (ca' + c'a)\cos B - (ab' + a'b)\cos C = 0,$$

or $$a'(a - c\cos B - b\cos C) + \&c. = 0.$$

But this is identically satisfied if a, b, c are the sides of the triangle of reference, or if $ax + by + cz = 0$ is the line at infinity.

[Mr. THOMSON sends this solution accompanied by the statement that he is unable to give a *direct* solution of the question.]

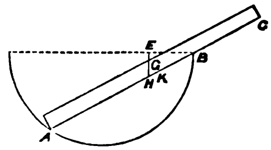
4651. (By A. MARTIN, M.A.)—A cylindric rod, of length $2a$ and radius r, rests with one end in contact with the concave surface of a fixed hemispherical bowl, radius r, and passes over the rim; find the inclination of the rod to the horizon.

Solution by the PROPOSER.

Let A be the point where the lower end of the rod rests against the concave surface of the bowl, B where it rests on the edge of the bowl, and G its centre of gravity.

When the rod is at rest its centre of gravity will be the lowest possible.

Let θ be the inclination of the rod to the horizon.



Then $KB = 2R\cos\theta - a$, $GH = r\sec\theta$,

$$HK = r\tan\theta,\quad HB = 2R\cos\theta + r\tan\theta - a,$$

$$EH = (2R\cos\theta + r\tan\theta - a)\sin\theta,$$

$$u = EG = (2R\cos\theta + r\tan\theta - a)\sin\theta - r\sec\theta,$$
$$= R\sin 2\theta + r\sin\theta\tan\theta - a\sin\theta - r\sec\theta = \text{a max.};$$

therefore $$\frac{du}{d\theta} = 2R\cos 2\theta + r\sin\theta - a\cos\theta = 0;$$

therefore $$2R\cos 2\theta = a\cos\theta - r\sin\theta;$$

hence we have

$$16R^2\sin^4\theta - 8Rr\sin^3\theta - (16R^2 - r^2 - a^2)\sin^2\theta + 4Rr\sin\theta = -(4R^2 - a^2),$$

an equation of the fourth degree for the determination of θ.

5956. (By Professor TOWNSEND, F.R.S.)—A slender uniform circular ring being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere in its space; show that the attraction:—

(*a*) For all positions of the particle, passes in direction through the vertex of the cone which envelopes along the circumference of the ring its sphere of connection with the position of the particle.

(*b*) For all positions of the particle, varies in magnitude directly as the distance from the centre, and inversely as the square of the product of the greatest and least distances from the circumference of the ring.

(*c*) For positions of the particle limited to a definite sphere passing through the circumference of the ring, varies in magnitude directly as the distance from the fixed centre through which it then passes, and inversely as the square of the distance from the plane of the ring.

Solution by W. J. C. SHARP, M.A.; Prof. MATZ, M.A.; *and others.*

Let the centre of the ring be taken as the origin, and let (r, θ) be the polar coordinates of the projection of the particle on the plane of the ring, and h its perpendicular distance from this plane. Then, it appears, as in Professor TOWNSEND's paper in the *Quarterly Journal* (V_{-3} being the potential) that $dV_{-3} = -\frac{1}{2}\frac{dm}{u^2} = -\frac{1}{4}\frac{m}{\pi}\,\frac{d\theta}{a^2+h^2+r^2+2ar\cos\theta}$,

therefore $$V_{-3} = -\tfrac{1}{2}m\,.\left\{(a^2+h^2+r^2)^2 - 4a^2r^2\right\}^{-\frac{1}{2}}.$$

Hence the forces along h and r are respectively

$$\frac{dV_{-3}}{dh} = +fm\,\frac{(a^2+h^2+r^2)\,h}{\left\{(a^2+h^2+r^2)^2-4a^2r^2\right\}^{\frac{3}{2}}},$$

and $$\frac{dV_{-3}}{dr} = +fm\,\frac{(h^2+r^2-a^2)\,r}{\left\{(a^2+h^2+r^2)^2-4a^2r^2\right\}^{\frac{3}{2}}}.$$

Now let p be the distance of the centre of sphere from plane of ring

$$a^2+p^2 = (h+p)^2+r^2;$$

hence tangent of the angle which the direction makes with

$$r = \frac{dV_{-3}}{ah} \div \frac{dV_{-3}}{ar} = \frac{(h^2+r^2+a^2)\,h}{(h^2+r^2-a^2)\,r} = \frac{pb-a^2}{pr},$$

which proves (a).

Again, $$\left\{\left(\frac{dV_{-3}}{dh}\right)^2+\left(\frac{dV_{-3}}{dr}\right)^2\right\}^{\frac{1}{2}} = -fm\,\frac{(h^2+r^2)^{\frac{1}{2}}}{(a^2+h^2+r^2)^2-4a^2r^2},$$

which proves (b); and this $= +fm\,\dfrac{(h^2+r^2)^{\frac{1}{2}}}{4b^2\,(p^2+a^2)}$, which proves ($c$).

6036. (By Prof. Minchin, M.A.)—Assuming the obvious result that the surface-integral of normal attraction over any closed surface is zero or $4\pi\times$ internal mass, write down at once (without any transformation) the polar differential equation for the Potential; and hence show instantaneously that, if $f(x, y, z)$ is a function which satisfies Laplace's equation, the function $\dfrac{a}{r}f\left(\dfrac{a^2x}{r^2}, \dfrac{a^2y}{r^2}, \dfrac{a^2z}{r^2}\right)$ will also satisfy it (a is constant; $r^2 \equiv x^2+y^2+z^2$). [The terrible process of transformation by which this is usually proved is thus completely avoided, and the result obtained without any trouble whatever.]

Solution by T. R. Terry, M.A.; Belle Easton; *and others.*

We have $\displaystyle\iint \frac{dV}{dn}\,dS = -4\pi M$. Applying this to the elementary volume in polar coordinates, we obtain as follows:—

Sum of integral for two faces for which r is const. $\Big\} = \left(r^2\dfrac{d^2V}{dr^2}+2r\dfrac{dV}{dr}\right)\sin\theta\,dr\,d\theta\,d\phi$,

...... for which θ is constant $= dr\,d\theta\,d\phi\left(\cos\theta\,\dfrac{dV}{d\theta}+\sin\theta\,\dfrac{d^2V}{d\theta^2}\right)$,

...... for which ϕ is constant $= dr\,d\theta\,d\phi\,\dfrac{1}{\sin\theta}\,\dfrac{d^2V}{d\phi^2}$,

and $$M = \rho\,dr\,d\theta\,d\phi\,r^2\sin\theta;$$

therefore $$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}+\frac{\cot\theta}{r^2}\frac{dV}{d\theta}+\frac{1}{r^2}\frac{d^2V}{d\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2}+4\pi\rho = 0.$$

Solution by R. Graham, M.A.; J. A. Steggal, B.A.; *and others.*

Let the quartic be $(a_0 a_1 a_2 a_3 a_4)(x1)^4 = 0$, with roots $\alpha, \beta, \gamma, \delta$; and let the reducing cubic be $y^3 - \frac{I}{a_0^2} y + \frac{2J}{a_0^3} = 0$, with roots t_1, t_2, t_3; then

$$t_1 = \tfrac{1}{4}(\beta\gamma + \alpha\delta) - \frac{a_2}{a_0}, \text{ \&c.,}$$

$$[x-(\beta-\gamma)^2][x-(\alpha-\delta)^2] = x^2 - \left(\alpha^2+\beta^2+\gamma^2+\delta^2-4t_1-\frac{4a_2}{a_0}\right)x + 4(t_2-t_3)^2$$

$$= x^2 - 4\left(\frac{4H}{a_0^2} - t_1\right)x + \frac{4I}{a_0^2} - 12t_2t_3 = 4\left\{\lambda + t_1x + \frac{6J}{a_0^3t_1}\right\},$$

where $$4\lambda = x^2 - \frac{16H}{a_0^2}x + \frac{4I}{a_0^2}.$$

Since $t_1+t_2+t_3 = 0$, $t_2t_3+t_3t_1+t_1t_2 = -\frac{2I}{a_0^2}$, and $t_1t_2t_3 = -\frac{2J}{a_0^3}$.

Hence the equation of the differences is

$$\left(\lambda + t_1x + \frac{6J}{a_0^3t_1}\right)\left(\lambda + t_2x + \frac{6J}{a_0^3t_2}\right)\left(\lambda + t_3x + \frac{6J}{a_0^3t_3}\right) = 0, \text{ or}$$

$$\lambda^3 + \frac{3I}{0^2}\lambda^2 - \left(\frac{I}{a_0^2}x^2 + \frac{18J}{a_0^3}x\right)\lambda - \left(\frac{2J}{a_0^3}x^3 + \frac{3I^2}{a_0^4}x + \frac{36IJ}{a_0^5}x + \frac{108J^2}{a_0^6}\right) = 0.$$

6042. (By Professor Burat.)—Aux centres de gravité $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ des faces d'un tétraèdre on applique de forces $\pi_1, \pi_2, \pi_3, \pi_4$ normales à ces faces et proportionnelles à leurs surfaces. Le tétraèdre, est-il en équilibre? Si oui, le démontrer.

I. *Solution by* T. R. Terry, M.A.; H. R. Robson, B.A.; *and others.*

It is well known that the tetrahedron is in equilibrium. Of the many proofs that have been given, one of the neatest is obtained from the consideration of a fluid in equilibrium under the action of no internal forces. The pressure being uniform throughout, if we consider a portion of the fluid of the shape of the tetrahedron to become solidified, we obtain the required result.

II. *Solution by* F. D. Thomson, M.A.; J. S. Jenkins, B.A.; *and others.*

Let OABC be the tetrahedron. Then, assuming that all the four forces are directed inwards or all outwards, the four forces may be represented by V(OA.OB), V(OB.OC), V(OC.OA), V(AC.AB).

Let α, β, γ denote the vectors OA, OB, OC.

Now the force perpendicular to the face OAB through the centre of gravity of OAB is equivalent to three equal forces of one-third the magnitude acting in the same direction at th[illegible] ers O, A, B. Resolve all the four given forces in the same wa[illegible]

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Solution by the Rev. J. L. Kitchin, M.A.; E. Rutter; *and others.*

If $x = a_0\lambda^n + a_{n-1}\lambda^{n-1} + \&c.$, $y = b_0\lambda^n + b_{n-1}\lambda^{n-1} + \&c.$, $z = \&c.$, represent a straight line $Ax + By + Cz = 0$, then

$$\lambda^n (Aa_0 + Bb_0 + Cc_0) + \lambda^{n-1} (Aa_1 + Bb_1 + Cc_1) + \&c. = 0,$$

an equation which involves (since λ is a variable)

$$Aa_0 + Bb_0 + Cc_0 = 0,\ Aa_1 + Bb_1 + Cc_1 = 0,\ Aa_2 + Bb_2 + Cc_2 = 0,\ \&c., \&c.,$$

or &c., as in the Question.

5999. (By Donald McAlister, B.A., B.Sc.)—Prove that the ratio of the diameter of curvature to the normal chord at any point on an ellipse, is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2}$, where (x, y) is the extremity of the corresponding conjugate diameter.

Solution by R. Graham, M.A.; R. Warren, M.A.; *and others.*

If n, n' be the lengths of normals drawn at the extremities of any chord of an ellipse, p, p' the perpendiculars from the centre on the tangents at the extremity of the chord, and b_1 the semi-diameter parallel to the chord; then, by a well-known theorem, $np + n'p' = 2b_1^2$.

Supposing the chord of the ellipse to be a normal chord, we have

$$n' = 0, \text{ and } n = \frac{2b_1^2}{p}, \text{ also } 2\rho = \frac{2b'^2}{p};$$

therefore

$$\frac{2\rho}{n} = b'^2 \cdot \frac{1}{b_1^2} = b'^2\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{a'^2}\right)$$

$$= (x^2 + y^2)\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 1$$

$$= \frac{x^2}{b^2} + \frac{y^2}{a^2} + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \frac{x^2}{b^2} + \frac{y^2}{a^2}.$$

5769. (By J. C. Malet, M.A.)—(1) A variable plane cuts a given volume from a given ellipsoid, prove that it envelops a similar ellipsoid. (2) In the same case, prove that the volume of cone of which the base is the plane section of the ellipsoid, and the vertex the centre, is also constant. (3) If however the volume of the cone be given, and not that of the section of the ellipsoid, prove that the plane touches one of *three* similar ellipsoids.

Solution by Professor NASH, M.A.; F. WERTSCH; *and others.*

Let $lx+my+nz=p$ be a plane cutting off, from the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1, \text{ a volume } (\tfrac{1}{2}-k)\,\tfrac{4}{3}\pi abc;$$

then we have $\quad k\,\tfrac{4}{3}\pi abc = \displaystyle\int_0^p \frac{\pi abc}{(l^2a^2+m^2b^2+n^2c^2)^{\frac{1}{2}}}\left(1-\frac{2^2}{a^2}\right)dr;$

where a is length of perpendicular from centre on the tangent plane parallel to the cutting plane,

therefore $\quad a^2 = l^2a^2+m^2b^2+n^2c^2$ and $4k = \dfrac{3a^2p-p^3}{a^3}$ ....(1);

therefore $lx+my+nr=p$ is a tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\lambda^2,$$

where λ is a root of the equation (1) in $\dfrac{p}{a}$, *i.e.*, $\lambda^3-3\lambda+4k=0$ ....(2).

With the same notation the volume of the cone whose base is the plane section and vertex the centre of the ellipsoid is evidently

$$\frac{\pi abcp}{3a}\left(1-\frac{p^2}{a^2}\right) \text{ or } \frac{\pi abc}{3}(\lambda-\lambda^3),$$

which, by (2), is constant, if λ be so.

If either of the two volumes be given we shall have a cubic equation to determine λ, and therefore, three enveloped ellipsoids.

6052. (By the Rev. W. ALLEN WHITWORTH, M.A.)—Find a fraction with a prime denominator, which when expressed decimally has a recurring period consisting of the ten digits. Show that there is only one possible denominator and 240 different numerators.

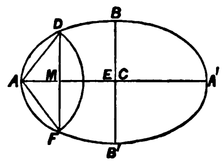
Solution by G. HEPPEL, M.A.; G. TURRIFF, M.A.; *and others.*

Since the factors of 9999999999 are $3^2\times 11\times 41\times 271\times 9091$, and 11 produces a period of two figures, and 41 and 271 each produce a period of five figures; it follows that 9091 is the sole possible prime denominator.

If A be such an arrangement of the ten digits as to form a period corresponding to a fraction $\dfrac{P}{9091}$, then A must be divisible by 99999, and also by 11. The condition of its being divisible by 99999 is that the sums of the 1st and 6th, 2nd and 7th digits, &c., shall be severally equal to 9. The condition of its being divisible by 11 is that the sums of the digits in the odd and even places shall be equal, one to 17, the other to 28. Hence, out of the pairs 4, 5; 3, 6; 2, 7; 1, 8; 0, 9; we have to take one out of each so that the sum of the five may be 17. The only possible combination is 43280. Hence one form of A is 4139258607, and the digits in the odd places may be permuted 120 ways. Lastly, the set whose sum is 28 may be put first, thus giving 240 arrangements, and 240 numerators.

5799. (By Professor MATZ, M.A.)—From one end of the major axis of an ellipse whose minor axis is *unknown*, a circle is drawn at random, but so as to cut the ellipse; find (1) the average area common to the circle and ellipse; (2) the average area common to the circle and ellipse if the minor axis be *known*.

Solution by E. B. SEITZ.



Let ABA′B′ be the ellipse, DEF the arc of the circle within the ellipse. Let CA $= a$, CB $= b$, AM $= x$, DM $= y$, AD $= r$, the common area AEDF $=$ S, $a - x = a\cos\theta$, and let $\Delta =$ the average of the common area when b is known, and $\Delta_1 =$ the average when b is unknown. Then $y = b\sin\theta$, $r = 2a\sin\frac{1}{2}\theta\,(1-e^2\cos^2\frac{1}{2}\theta)^{\frac{1}{2}}$,

$$\angle\,\mathrm{DAE} = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right),$$

the elliptic segment DAF $= ab\,(\theta - \sin\theta\cos\theta)$, and the circular segment

$$\mathrm{DEF} = r^2\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right) - ab\,(\sin\theta - \sin\theta\cos\theta);$$

therefore $$\mathrm{S} = ab\,(\theta - \sin\theta\cos\theta) + r^2\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right).$$

Hence, since the limits of r are 0 and $2a$, and those of θ are 0 and π, we have

$$\Delta = \frac{1}{2a}\int_0^{2a}\mathrm{S}\,dr = \frac{1}{2a}\left[abr\,(\theta-\sin\theta) + \tfrac{1}{3}r^3\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right)\right]_0^{2a}$$
$$- \tfrac{4}{3}ab\int_0^{\pi}(1-e^2\cos^2\tfrac{1}{2}\theta)^{\frac{1}{2}}\sin^3\tfrac{1}{2}\theta\,d\theta$$
$$= \left\{\pi - \left(\frac{4e^2-1}{3e^3}\right)\sin^{-1}e - \frac{b}{a}\left(\frac{2e^2+1}{3e^2}\right)\right\}ab.$$

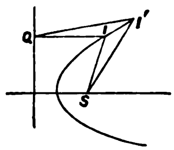
For Δ_1 the limits of b are 0 and a; hence we have

$$\Delta_1 = \frac{1}{a}\int_0^a \Delta\,db$$
$$= \tfrac{1}{2}\pi a^2 - \frac{1}{3a}\int_0^a\left\{2b^2 - a^2 + \frac{a^4}{a^2-b^2} + \left[\frac{4a^2b}{(a^2-b^2)^{\frac{3}{2}}} - \frac{a^4b}{(a^2-b^2)^{\frac{1}{2}}}\right]\sin^{-1}e\right\}db$$
$$= \left(\frac{16}{9} - \frac{\pi}{3}\right)a^2.$$

5952. (By W. GALLATLY, B.A.)—Find geometrically the envelop of the system of circles represented by

$$(x - am^2)^2 + (y - 2am)^2 = a^2\,(1+m^2)^2.$$

Solution by the PROPOSER ; JOSIAH SCOTT, M.A. ; *and others.*



This equation represents a circle whose centre is any point P on the parabola $y^2 = 4ax$ and radius $=$ SP.

Take consecutive centres P, P′ on the parabola: the consecutive circles cut in Q, where $PQ = SP : P'Q = SI'$. Therefore the envelop is the directrix.

5632. (By Professor WOLSTENHOLME, M.A.)—Prove that the envelop of the radical axis of circles of curvature drawn at the ends of a focal chord of a parabola (the origin being the foot of the directrix) is the third class quartic

$$\left(x-\frac{2y}{\sqrt{3}}\right)^{-\frac{1}{2}}+\left(x+\frac{2y}{\sqrt{3}}\right)^{-\frac{1}{2}}+\left(\frac{2x-9a}{8}\right)^{-\frac{1}{2}}=0,$$

I. *Solution by* D. EDWARDES ; J. HEPPEL, M.A. ; *and others.*

Let the coordinates of one end of the chord be $(am_1^2,\ 2am_1)$, then the coordinates of other end will be $\dfrac{a}{m_1^2},\ -\dfrac{2a}{m_1}$. The circles of curvature therefore are

$$\{x-a(3m_1^2+2)\}^2+(y+2am_1^3)^2=4a^2(1+m_1^2)^3$$

or

$$\left\{x-a\left(\frac{3}{m_1^2}+2\right)\right\}^2+\left(y-\frac{2a}{m_1^3}\right)^2=4a^2\left(1+\frac{1}{m_1^2}\right)^3;$$

therefore the radical axis $\left(\text{subtracting and dividing out by } m_1+\dfrac{1}{m_1}\right)$ is

$$-6x\left(m_1-\frac{1}{m_1}\right)+4y\left(m_1^2+\frac{1}{m_1^2}-1\right)=3a\left(m_1-\frac{1}{m_1}\right)\left(m_1^2+\frac{1}{m_1^2}\right)$$

or

$$-6kx+4y(k^2+1)=3ak(k^2+2)\quad\text{if } k\equiv m_1-\frac{1}{m_1}.$$

Differentiating with respect to k, we have $-6x+8yk=3a(3k^2+2)$.
Transferring to $(-a_1,\ 0)$, these last two equations become

$$-6kx+4y(k^2+1)=3ak^3,\quad -6x+8yk=9ak^2.$$

Eliminating k, we have $\dfrac{y\sqrt{3}\,(2x-9a)}{\{3x(2x-9a)-2(3x^2-4y^2)\}^{\frac{1}{2}}}=\sqrt{2}\,(3x^2-4y^2$,

or

$$\frac{(Q^2-P^2)R^2}{\{R^2(P^2+Q^2)-4P^2Q^2\}^{\frac{1}{2}}}=4PQ,$$

where

$$P^2\equiv x-\frac{2y}{\sqrt{3}},\quad Q^2\equiv x+\frac{2y}{\sqrt{3}},\quad R^2\equiv 2x-9a;$$

whence

$$R=\pm\frac{\sqrt{8}\,.\,PQ(P\pm Q)}{P^2-Q^2}\quad\text{or}\quad\frac{1}{P}\mp\frac{1}{Q}=\pm\frac{2\sqrt{2}}{R}.$$

II. *Solution by the* PROPOSER.

The circle of curvature at the point $\left(\frac{a}{m^2}-a,\ \frac{2a}{m}\right)$ (focus origin)

is $$m^2(x^2+y^2-2ax)-6ax+\frac{4ay}{m}-3a^2\left(m+\frac{1}{m}\right)^2=0,$$

and at the other end of the focal chord $(am^2-a,\ -2am)$ is

$$\frac{1}{m^2}(x^2+y^2-2ax)-6ax-4aym-3a^2\left(\frac{1}{m}+m\right)^2=0.$$

The equation of the radical axis is, therefore,

$$6ax\left(m^2-\frac{1}{m^2}\right)+4ay\left(m^3+\frac{1}{m^3}\right)+3a^2\left(m+\frac{1}{m}\right)^2\left(m^2-\frac{1}{m^2}\right)=0,$$

or $$6x\left(m-\frac{1}{m}\right)+4y\left(m^2-1+\frac{1}{m^2}\right)+3a\left(m+\frac{1}{m}\right)^2\left(m-\frac{1}{m}\right)=0,$$

or, if $m-\frac{1}{m}=\lambda$, $\quad 6\lambda x+4y(\lambda^2+1)+3a\lambda(\lambda^2+4)=0,$

or $$3a\lambda^3+4y\lambda^2+6(x+2a)\lambda+4y=0;$$

whence, for the envelop, $\quad 9a\lambda^2+8\lambda y+6(x+2a)=0,$

and, for the cusps, $\quad 9a\lambda+4y=0.$

These equations give the cusps $\lambda=0$, $x+2a=0$, $y=0$; $\lambda=\mp\sqrt{3}$, $x+2a=\frac{9}{2}a$, $y=\pm\frac{9}{4}\sqrt{3}a$; so that the three lines joining the cusps two and two are, moving the origin to the foot of the directrix, $2x=9a$, $2y\pm x\sqrt{3}=0$; also for any point on the curve

$$6\lambda x+4(1+\lambda^2)y=-3a\lambda^3,\quad 6x+8\lambda y=-9a\lambda^2,$$

so that $$2y=\frac{3a\lambda^3}{1-\lambda^2},\qquad 2x=\frac{a\lambda^2(3+\lambda^2)}{\lambda^2-1}.$$

Hence $$2x-9a=a\,\frac{(3-\lambda^2)^2}{\lambda^2-1},\quad 4y-2\sqrt{3}x=a\,\frac{\sqrt{3}\lambda^2(\lambda+\sqrt{3})^2}{1-\lambda^2},$$

and $$2\sqrt{3}x+4y=a\,\frac{\sqrt{3}\lambda^2(\lambda-\sqrt{3})^2}{\lambda^2-1};$$

so that $$\left(2x+\frac{4y}{\sqrt{3}}\right)^{-\frac{1}{3}}=\left(\frac{(\lambda^2-1)}{a}\right)^{\frac{1}{3}}\frac{1}{\lambda(\lambda-\sqrt{3})},$$

$$\left(2x-\frac{4y}{\sqrt{3}}\right)^{-\frac{1}{3}}=\left(\frac{(\lambda^2-1)}{a}\right)\frac{1}{\lambda(\lambda+\sqrt{3})}$$

Distance between centres of curvature at ends of a focal chord is

$$a\left\{9\left(m^2-\frac{1}{m^2}\right)^2+4\left(m^3+\frac{1}{m^3}\right)^2\right\}^{\frac{1}{2}}$$

$$=a\left(m+\frac{1}{m}\right)\left\{9\left(m-\frac{1}{m}\right)^2+4\left(m^2-1+\frac{1}{m^2}\right)^2\right\}^{\frac{1}{2}},$$

and the difference of the radii is

$$2a\left(m+\frac{1}{m}\right)^{\frac{3}{2}}\left(m^{\frac{3}{2}}-\frac{1}{m^{\frac{3}{2}}}\right),$$

so that, when the circles touch, we have

$$9\left(m-\frac{1}{m}\right)^2+4\left(m^2-1+\frac{1}{m^2}\right)^2=4\left(m+\frac{1}{m}\right)\left(m^3-2+\frac{1}{m^3}\right),$$

or, if $m+\frac{1}{m}=\lambda$, $\quad 9(\lambda^2-4)+4(\lambda^2-3)^2=4\lambda\left\{\lambda(\lambda^2-3)-2\right\}$,

giving $\lambda=0$ or 8. The latter gives the only real solutions $\lambda=\frac{4\pm\sqrt{7}}{3}$; and, as before, where the circles meet,

$$(x^2+y^2-4ax+3a^2)\left(m-\frac{1}{m}\right)+4ay=0,$$

giving $$x^2+y^2-4ax-\frac{6ay}{\sqrt{7}}+3a^2=0,$$

and $$\left(m^2+\frac{1}{m^2}\right)(x^2+y^2-4ax-3a^2)-12ax-4ay\left(m-\frac{1}{m}\right)=0,$$

giving $$\frac{46}{9}(x^2+y^2-4ax-3a^2)-12ax+\frac{8\sqrt{7}}{3}ay=0,$$

or $$x^2+y^2-\frac{146}{23}ax+\frac{12\sqrt{7}}{23}ay-3a^2=0;$$

whence $$37y-9\sqrt{7}x-23\sqrt{7}a=0,$$

which touches both circles (and the curve) in the point $\left(\frac{13}{11}a,\ \frac{10\sqrt{7}}{11}\right)$. The distances of this point from the forms of the parabola, the acnode of the curve, and the centre of inversion, are $\frac{8}{\sqrt{11}}a$, $\frac{10}{\sqrt{11}}a$, $\frac{16}{\sqrt{11}}a$; and its distance from the double tangent $x-y\sqrt{3}+3a=0$ is $\frac{23-5\sqrt{21}}{11}a$, or $\frac{2a}{253}$ nearly.

The common tangent at this point makes with the double tangent an angle $\tan^{-1}(\cdot 0483)$ or $2°\,45'\,49''\cdot 3$. The angle between a double tangent and the ordinary tangent drawn through the centre of inversion is $4'\,26''\cdot 6$, so that it would be very difficult to shew the true form of the curve unless the scale be very large. Hence we have

$$\left(2x+\frac{4y}{\sqrt{3}}\right)^{-\frac{1}{2}}+\left(2x-\frac{4y}{\sqrt{3}}\right)^{-\frac{1}{2}}=\frac{2}{\lambda^2-3}\left(\frac{\lambda^2-1}{a}\right)^{\frac{1}{2}}=\pm 2(2x-9a)^{-\frac{1}{2}},$$

or $$\left(x+\frac{2y}{\sqrt{3}}\right)^{-\frac{1}{2}}+\left(x-\frac{2y}{\sqrt{3}}\right)^{-\frac{1}{2}}+\left(\frac{2x-9a}{8}\right)^{-\frac{1}{2}}=0.$$

The rationalized equation is the discriminant of

$$(3a,\ \tfrac{4}{3}y,\ 2x,\ 4y \between \lambda,\ 1)^3=0;$$

or $$64y^4-9y^2(4x^2+36ax-27a^2)+162ax^3=0.$$

All four values of y will be real if $2x>9a$; two values if x be negative; and none if x lie between these limits. The rectilinear asymptotes are

found by putting $\lambda = \pm 1$, since x and y are then infinite, so that we get the two

$$8y \pm 3(2x+a) = 0,$$

meeting the curve again when $\lambda = +1 \pm \sqrt{2}$ or $-1 \pm \sqrt{2}$. There is also a parabolic asymptote $2y^2 = 9a(x-a)$, having six-pointic contact at infinity and meeting the curve again when $x = 5a$. The curve is rather troublesome to draw except on a large scale, but sufficient data have been given if any one should feel inclined to take the trouble.

6018. (By ALEXANDER MACFARLANE, D.Sc., F.R.S.E.)—In a certain town, those citizens who have an attribute A, and either an attribute B or an attribute C, have an attribute D, and *vice versâ*. The probability of a given citizen (1) having the attribute B is p_1, (2) having the attribute C is p_2, (3) having the attributes B and C is p_3, (4) having the attributes B and D is p_4, (5) having the attributes C and D is p_5. What is the probability of a given citizen having the attribute A?

I. *Solution by* H. C. ROBSON, B.A.

Let x be the probability of a citizen having an attribute A. Then, of n citizens (n being large) xn have A, and $(1-x)n$ have not A. Of the xn, p_1xn have B: p_2xn have C; p_3xn have B and C; all these have D; and $(1-p_1-p_2+p_3)xn$ have neither B nor C.

Of the $(1-x)n$ which have not A, $p_1(1-x)n$ have B; $p_2(1-x)n$ have C; $p_3(1-x)n$ have B and C; $(1-p_1-p_2+p_3)(1-x)n$ have neither B nor C; and none of these have D, for if they had they would have A and either B or C.

p_4n = no. that have B and D $= p_1xn$;
p_5n = no. that have C and D $= p_2xn$; therefore $x = \dfrac{p_4}{p_1} = \dfrac{p_5}{p_2}$.

II. *Solution by the* PROPOSER.

Let a denote having the attribute A, b denote having the attribute B, and so on. Then the given relation among the attributes is expressed by the equation $a(b+c) = d$, therefore

$$a = \frac{d}{b+c} = \tfrac{1}{2}bcd + \tfrac{0}{2}bc(1-d) + \tfrac{1}{1}b(1-c)d + \tfrac{0}{1}b(1-c)(1-d) + \tfrac{1}{1}(1-b)cd$$
$$+ \tfrac{0}{1}(1-b)c(1-d) + \tfrac{1}{0}(1-b)(1-c)d + \tfrac{0}{0}(1-b)(1-c)(1-d)$$

therefore $\qquad bcd = 0 \quad$ and $\quad (1-b)(1-c)d = 0.$

Hence $\quad a = bd + cd + \tfrac{0}{0}(1-b)(1-c) = bd + cd + \tfrac{0}{0}(1-b-c+bc)$

therefore $\qquad a' = (bd)' + (\quad)' + \tfrac{0}{0}[1' - b' - c' + (bc)'],$

(where the accent denotes that the arithmetical value of the symbol is taken,)

or $\qquad a' = p_4 + p_5 + \tfrac{0}{0}(1-p_1-p_2+p_3)$, by the given data.

Hence a' is greater than p_4+p_5, and less than $p_4+p_5+1-p_1-p_2+p_3$.

Cor. 1.—Let $b=1,\ \ c=0,\ \ d=1.$

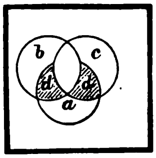
Then $p_4=1,\ \ p_5=0,\ \ p_3=0,$

therefore $a' \not< 1$ and $\not> 1+1-1$, therefore $a'=1$.

Similarly, if $b=0,\ \ c=1,\ \ d=1;$ then $a'=1$.

Cor. 2.—Let $b=0,\ c=0,\ d=0$; then $a'=\frac{0}{0}$; that is, may be any fraction between 0 and 1.

The annexed diagram is constructed to satisfy in the most general manner the given relation among the attributes; and it will be observed to satisfy each of the deduced equations.



Those citizens who have the attribute B are represented by the circle b, those who have the attribute C by the circle c, those who have the attribute A by the incomplete circle a, and those who have the attribute D by the two shaded portions.

I deduce these conclusions by means of principles established in my work entitled the *Algebra of Logic.* I hold that the principles there enuntiated constitute the foundations of an Algebra which absorbs the ordinary theories of Necessity and Probability and also the methods discovered by Boole.

In the foregoing solution Mr. Robson treats the question according to the ordinary theory of probability, and introduces the ordinary assumptions. It is not given that the number of citizens is large; nor is any relation among the attributes given, further than what is involved in the relation stated. The assumption is not warranted that the fraction of those citizens having the attribute A which have the attribute B is equal to the fraction of the whole number of citizens which have the attribute B.

III. *Solution by* C. J. Monro, M.A.

Mr. Robson's solution requires that $p_5 : p_2 = p_4 : p_1$. This relation is not a necessary one, as it is that $p_1-p_3,\ p_2-p_3,\ p_3-(p_1+p_2-1)$ shall not be negative; and it is just the gist of the question that no such relation is given.

In Mr. Jevons' notation, we have

$$A(B+C) = A(B+C)D, \text{ and, ``vice versa,'' } D = DA(B+C),$$

therefore $$D = A(B+C);$$

whence $BD = BA(B+C), = AB$, and $CD = CA(B+C), = CA$.

So the question may be stated without D and is this;—Among the probabilities of the simple and compound attributes

$$A,\ B,\ C,\ BC,\ CA,\ AB,$$

required the first, given the rest. As Mr. Macfarlane has distributed the suffixes unsymmetrically, I write these probabilities

$$a,\ b,\ c,\ l,\ m,\ n.$$

Observe that, in Boole, p. 334, the question is,—Among the same probabilities, required the last, given the rest. But the resulting relation is not the same in the two problems, being unsymmetrical in both.

Let the corresponding logical symbols be

$$x,\ y,\ z,\ u,\ v,\ w, \text{ where } u = yz,\ v = zx,\ w = xy \quad \ldots\ldots\ldots\ldots(1),$$

Then we have to express x as a logical function of y, z, u, v, w. The functions u, v, w are so simple that Boole's general method is unnecessary. For, since it is evident (negative elements being distinguished by accents) that $uvw = xyz$, $u'vw = 0$, $uv'w' = yz(z'+x')(x'+y')$, $= x'yz$,

and $u'v'w' = (y'+z')(z'+x')(x'+y')$, $= x'y'z' + y'z' + z'x' + x'y'$,

$$= x'y'z' + xy'z' + x'yz' + x'y'z;$$

it follows that $(x+x')(y+y')(z+z')(u+u')(v+v')(w+w')$ is, in virtue of the conditions (1),

$$\begin{aligned} &xyzuvw + x'yzuv'w' + xy'zu'vw' + xyz'u'v'w \\ &\qquad + x'y'z'u'v'w' + xy'z'u'v'w' + x'yz\,u'v'w' + x'y'zu'v'w', \\ &= x\,(yzuvw + y'zu'vw' + yz'u'v'w) \\ &\qquad + x'\,(yzuv'w' + y'zu'v'w' + yz'u'v'w') + (x+x')\,y'z'u'v'w'; \end{aligned}$$

whence, even by common sense, if $\frac{0}{0}$ represent indeterminate logical quantity,

$$\begin{aligned} x &= yzuvw + y'zu'vw' + yz'u'v'w \\ &\qquad + 0\,(yzuv'w' + y'zu'v'w' + yz'u'v'w') + \tfrac{0}{0}\,y'z'u'v'w'; \end{aligned}$$

terms having other coefficients being already eliminated. Thus, passing from logic to arithmetic, we have

$$\begin{aligned} &yzuvw + yzuv'w' + y'zu'vw' + yz'u'v'w + y'zu'v'w' + yz'u'v'w' + y'z'u'v'w' = \mathrm{V}, \\ &yzuvw + y'zu'vw' + yz'u'v'w + \tfrac{0}{0}y'z'u'v'w' = a\mathrm{V}, \\ &yzuvw + yzuv'w' + yz'u'v'w + yz'u'v'w' = b\mathrm{V}, \\ &yzuvw + yzuv'w' + y'zu'vw' + y'zu'v'w' = c\mathrm{V}, \\ &yzuvw + yzuv'w' = l\mathrm{V}, \\ &yzuvw + y'zu'vw' = m\mathrm{V}, \\ &yzuvw + yz'u'v'w = n\mathrm{V}. \end{aligned}$$

Divide both sides of these equations by $yzuvw$, which forms a term in each, and let

$$\frac{u'}{u},\ \frac{v'}{v},\ \frac{w'}{w},\ \frac{y'w'}{yw},\ \frac{z'v'}{zv},\ \frac{\mathrm{V}}{yzuvw},$$

be

$$\lambda,\ \mu,\ \nu,\ \rho,\ \sigma,\ \mathrm{R}.$$

Then the 4th, 5th, and 6th equations become

$$\lambda\mu\rho + \lambda\rho + \mu\nu + 1 = c\mathrm{R},\quad \mu\nu + 1 = l\mathrm{R},\quad \lambda\rho + 1 = m\mathrm{R};$$

whence $\qquad (m\mathrm{R}-1)\,\mu = (c-m-l)\,\mathrm{R}+1,$

and, similarly, $\qquad (n\mathrm{R}-1)\,\nu = (b-n-l)\,\mathrm{R}+1;$

and, substituting for μ and ν, we transform the 5th into a cubic for R,

$$(l\mathrm{R}-1)(m\mathrm{R}-1)(n\mathrm{R}-1) = \{(c-m-l)\,\mathrm{R}+1\}\{(b-n-l)\,\mathrm{R}+1\}\ldots.(2).$$

R found, the 1st equation would give $\lambda\rho\sigma$, thus,

$$\lambda\rho\sigma = \{l-(b+c-1)\}\,\mathrm{R}, \text{ whence } a = m+n-\frac{1}{\mathrm{R}} + \tfrac{0}{0}\{l-(b+c-1)\}.$$

In doubt as to the correct reading of the question, it seems hardly worth while to examine the cubic (2).

I have now seen the Proposer's *Algebra of Logic*, and, notwithstanding his

discussion of traditional testimony (pp. 151, 152, and *Reprint*, Vol. XXXII., pp. 18—19), I judge (from § 62 and ch. xx., xxii.) that he would exclude non-linear functions of given probabilities, thus reducing Quest. 6018 and most questions of probability to questions of "statistical" or "numerically definite" limiting conditions.

Here 6 quantities $a, b, \ldots n$ are sums of 4 or 2 out of 8 quantities, xyz, xyz', $xy'z$, ... taken arithmetically; and the sum of all 8 is 1. We can eliminate only 6 of these 8, but this in 28 ways, including 3 such results given, 3 all but given, and 22 pretty easily written down. The pair not eliminated always appear in their sum or difference; only sums give conditions; only 6 conditions involve a, namely:—

$$a-m = xyz'+xy'z', = z'x, \quad 1-c-a+m = x'yz'+x'y'z', = z'x',$$
$$a-n = xy'z+xy'z', = xy', \quad 1-a-b+n = x'y'z+x'y'z', = x'y',$$
$$a+l-m-n = xy'z'+x'yz, \quad 1-a-b-c+l+m+n = xyz+x'y'z'.$$

Therefore a is not less than m, n, $m+n-l$, and not greater than $1-c+m$, $1-b+n$, $1-b-c+l+m+n$, and is not otherwise limited.

6046. (By R. MOON, M.A.)—Prove that $\frac{1}{0}$ does not represent a quantity of infinite magnitude, but is simply an unmeaning symbol, incapable of any rational interpretation.

I. *Solution by* G. HEPPEL, M.A.; J. A. KEALY, M.A.; *and others.*

The usual mathematical definition of 0 is "that which is less than any assignable quantity"; and the usual mathematical definition of infinity is "that which is greater than any assignable quantity." If these definitions are recognised, it is obvious that $\frac{1}{0}$ must mean an infinite quantity.

II. *Solution by* H. L. ORCHARD, L.C.P.; F. PURSER, M.A.; *and others.*

If we suppose 0 to denote zero; then, (1) if $\frac{1}{0}$ = infinity $= \infty = \frac{\infty}{1}$; and therefore, by the property of fractions, $\frac{0}{1} = \frac{1}{\infty}$; therefore, adding unity to each side, we shall have $1 = \frac{1+\infty}{\infty}$; therefore $\infty = 1+\infty$, $1 = \infty - \infty = 0$, which is absurd.

(2) If $\frac{1}{0} = a$, where a is a finite quantity; then $\frac{0}{1} = \frac{1}{a}$, and, adding unity to each side, $1 = \frac{1+a}{a}$; therefore $a = 1+a$, therefore $1 = 0$.

Hence, again, if 0 denote zero, the symbol $\frac{1}{0}$ is absurd.

But, if 0 denote an *indefinitely small quantity*, h, say, then

$$\frac{1}{h} = \infty, \quad \frac{1+h}{1} = \frac{1+\infty}{\infty}, \quad \infty + h\infty = 1+\infty;$$

therefore $h\infty = 1$, which *is* possible.

III. *Solution by the* PROPOSER.

If n be a small quantity compared with the unit, then n^{-1} will be a large quantity, and if n be diminished indefinitely n^{-1} will be increased indefinitely. But this supposes that n is an actual magnitude. For, however small n may be, unity will be divisible by it, provided n is an actual magnitude. But 1 is not divisible by 0. The mere statement that 1 is divided by 0 is an absurdity. The words used can, in fact, have but one intelligible signification assigned to them, which is that 1 is not divided at all.

If, instead of interpreting n^{-1} as 1 divided by n, we interpret it, as we may do, as the ratio of the numbers of common parts contained by 1 and n respectively, the impossibility of attaching an intelligible signification to the symbol n^{-1} when $n = 0$ is equally conspicuous, for 0 has no parts, has nothing in common with any entity.

To disprove the contention of my opponent, it has not been necessary to impugn the definitions he has proposed. They are, however, not the less open to objection. The definition of infinity as "that which is greater than any assignable quantity," treats infinity as a definite entity, which it is not. We can, in fact, only conceive of infinity as a very large number, which we are at liberty to increase indefinitely when anything appears likely to be gained by doing so. Indefiniteness is its essential characteristic. The objection to the definition of 0, which must, I should think, strike most persons as somewhat artificial, is different.—If N be a large number, N^{-1} will be a small number, and if we increase the first indefinitely, we shall diminish the second indefinitely; and as we can assign no limit to the magnitude of N, so we can assign no limit to the smallness of N^{-1}, although all the while continuing to be an actual magnitude. Hence the definition of 0 as "that which is less than any assignable quantity" is no definition at all, since, however small the quantity assigned there will always be an infinite number of magnitudes inferior to it.

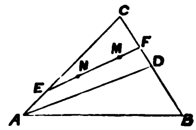
When the substitution of a particular value for one of the variables reduces a formula, or a portion of it, to the form $\frac{1}{0}$, it will be found, invariably, that the introduction of that value contradicts one or more of the assumed data of the problem: the absurdity of the result thus indicating a fallacy in the argument under which it was derived.

NOTE ON QUESTION 5783. *By the* EDITOR.

Mr. WOOLHOUSE remarks, in a letter, that Mr. WHITWORTH's solution to Question 5783 (*Reprint*, Vol. XXXI., pp. 50—52), is, in his opinion, most certainly correct. The solution by the Proposer seems to be fettered by an unnecessary restriction that the cards shall be arranged in p distinct groups, each of which shall contain one partition of every suit. Mr. WHITWORTH's solution properly only takes into consideration that the p partitions of each suit shall be severally separated from each other by the other cards. Mr. WHITWORTH, however, is not so fortunate in testing Mr. TEBAY's result for the case when $p = 1$. In this case it comes to identically the same as his own, which, as he says truly, is obviously right.

5957. (By Professor Crofton.)—Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions; show that (the triangle being unity) the mean value of the *greater* portion is $\frac{7}{12}+\frac{1}{6}\log 2$.

Solution by E. B. Seitz.



Let ABC be the triangle, D the middle point of BC, M and N two points such that EF, the line through them, meets AC and BC, and makes an angle with AC less than the angle A.

Let $CE = x$, $EM = y$, $MN = z$, $EF = y'$, $\angle CEF = \theta$, $\angle CAD = \beta$, area $CEF = v$, and

$$u = \frac{b^2 \sin C \sin\theta}{2\sin(\theta+C)};\ \text{then}\ y' = \frac{x\sin C}{\sin(\theta+C)},\quad v = \frac{x^2 \sin C\sin\theta}{2\sin(\theta+C)},$$

$$du = \frac{b^2\sin^2 C\,d\theta}{2\sin^2(\theta+C)},\quad dv = \frac{x\sin C\sin\theta\,dx}{\sin(\theta+C)};$$

an element of the triangle at M is $\sin\theta\,dy\,dx$, and at N it is $d\theta\,z\,dz$.

When θ is less than β, AEFB is the greater portion, while x varies from 0 to b. But when θ is greater than β, AEFB is the greater portion, while x varies from 0 to $\left[\frac{\sin(\theta+C)}{\sin C\sin\theta}\right]^{\frac{1}{2}} = x'$, and EFC is the greater portion, while x varies from x' to b. The limits of y are 0 and y', and those of z are 0 and y, and doubled. The values of u corresponding to the values 0, β, A of θ, are 0, $\frac{1}{2}$, 1; and those of v corresponding to the values 0, x', b of x, are 0, $\frac{1}{2}$, u. Hence, since the whole number of ways the two points can be taken is expressed by unity, we have

$$2\int_0^\beta\int_0^b\int_0^{y'}\int_0^y (1-v)\sin\theta\,d\theta\,dx\,dy\,z\,dz$$

$$+2\int_\beta^A\left\{\int_0^{x'}(1-v)\,dx+\int_{x'}^b v\,dx\right\}\int_0^{y'}\int_0^y \sin\theta\,d\theta\,dy\,z\,dz$$

$$=\tfrac{1}{3}\int_0^\beta\int_0^b(1-v)\sin\theta\,y'^3\,dx+\tfrac{1}{3}\int_\beta^A\left\{\int_0^{x'}(1-v)\,y'^3\,dx+\int_{x'}^b v y'^3\,dx\right\}\sin\theta\,d\theta$$

$$-\tfrac{2}{3}\int_0^{\frac{1}{2}}\int_0^u(1-v)\,u^{-1}\,du\,v\,dv+\tfrac{2}{3}\int_{\frac{1}{2}}^1\left\{\int_0^{\frac{1}{2}}(1-v)\,v\,dv+\int_{\frac{1}{2}}^u x^2\,dx\right\}u^{-1}\,du$$

$$=\tfrac{1}{9}\int_0^{\frac{1}{2}}(3-2u)\,u\,du+\tfrac{1}{36}\int_{\frac{1}{2}}^1(8u^2+u^{-1})\,du=\tfrac{7}{72}+\tfrac{1}{36}\log 2.$$

This result is the same as that for each of five other expressions similar to that above. Hence the mean value in question is $\frac{7}{12}+\frac{1}{6}\log 2$.

5990. (By Prof. DESBOVES.)—Dans tout triangle ABC, on a

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Solution by E. RUTTER; R. KNOWLES, B.A., L.C.P.; *and others.*

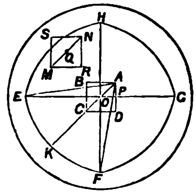
$$\Sigma \cos A = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

$$= 1 + \left(\frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \div \frac{a}{2 \sin A}\right) = 1 + \frac{r}{R}.$$

5725. (By E. B. SEITZ.)—If the line joining two points taken at random in the surface of a given circle be the diagonal of a square; show that the chance that the square lies wholly within the circle is $2 - 4\pi^{-1}$.

Solution by Prof. NASH, M.A.; *the* PROPOSER, *and others.*

Let MN be the line joining the random points M, N, MRNS the square, Q its centre, O the centre of the circle. Let the square move about the circle, so as to be within it, but in contact with it, the diagonal MN remaining parallel to itself; Q will describe the figure EFGH, whose boundary consists of the four equal arcs, EF, FG, GH, HE, whose centres are A, B, C, D, and radius equal to that of the given circle. Now while MN is given in length and direction, the area of the figure EFGH represents the number of favourable positions of the two points. Let $AE = r$, $MN = 2x$, $\angle AEP = \theta$, area $EFGH = u$, and $\phi =$ the angle which MN makes with some fixed line. Then we have



$$OA = x,\quad AP = \tfrac{1}{2}x\surd 2,\quad \angle EAK = AOP - AEP = \tfrac{1}{4}\pi - \theta,$$

$$\tfrac{1}{2}x\surd 2 = r \sin\theta,\quad EK = r \sin(\tfrac{1}{4}\pi - \theta),$$

area $EOF = r^2(\frac{1}{4}\pi - \theta) - rx \sin(\frac{1}{4}\pi - \theta) = r^2(\frac{1}{4}\pi - \theta - \sin\theta\cos\theta + \sin^2\theta)$,

and $$u = r^2(\pi - 4\theta - 4\sin\theta\cos\theta + 4\sin^2\theta).$$

An element of the circle at N is $4x\,dx\,d\phi$, or $8r^2 \sin\theta\cos\theta\,d\theta\,d\phi$; the limits of θ are 0 and $\frac{1}{4}\pi$; and of ϕ, 0 and 2π. Hence the chance in question is

$$p = \frac{1}{\pi^2 r^4}\int_0^{\frac{1}{4}\pi}\int_0^{2\pi} u \,.\, 8r^2 \sin\theta\cos\theta\,d\theta\,d\phi$$

$$= \frac{16}{\pi}\int_0^{\frac{1}{4}\pi} (\pi - 4\theta - 4\sin\theta\cos\theta + 4\sin^2\theta)\sin\theta\cos\theta\,d\theta = 2 - \frac{4}{\pi}.$$

5784. (By J. HAMMOND, M.A.)—If in the curve $xyz = a^3$ a chord be drawn parallel to the asymptote x; prove that the line joining the points,

where the tangents at the extremities of the chord cut the curve again, will also be parallel to x.

Solution by A. W. SCOTT, M.A.; J. L. MCKENZIE, M.A.; *and others.*

This is merely a particular case of the theorem that the tangents at three collinear points on any cubic meet the curve again in a collinear triad of points. Any chord parallel to x meets the curve in two finite points A, B, and in the infinite point I at which x is inflexional tangent. The tangents at A, B, and I meet the curve again in three collinear points. But the tangent at I is x, which meets the curve again in I; hence the tangents at A and B meet the curve in points on a line parallel to x.

5974. (By T. MITCHESON, B.A., L.C.P.)—Show that

$$13^{3n} - 648n^2 + 180n - 1$$

cubic inches are an exact number of cubic feet.

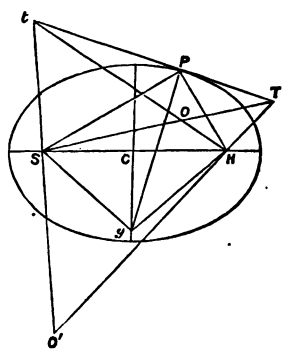
Solution by the Rev. DR. KNISELY; R. E. RILEY, B.A.; *and others.*

We have $\quad 13^{3n} = (1+12)^{3n} = 1 - 180n + 648n^2 + \text{M}\,(12^3)$;

hence $\quad 13^{3n} - 648n^2 + 180n^2 - 1 = \text{M}\,(12^3)$; therefore, &c.

5936. (By the Rev. T. J. SANDERSON, M.A.)—A pair of conjugate diameters meet a fixed tangent to an ellipse in T and t; prove that the lines joining T and t with the foci intersect on a fixed circle, and find its centre and radius.

Solution by A. W. SCOTT, M.A.; G. TURRIFF; *and others.*



Drawing the normal Pg, we have

$$\angle \text{SPT} = \angle \text{HPT},$$

and $\quad \text{PT} \cdot \text{P}t = \text{SP} \cdot \text{PH}$;

therefore the triangles SPT and HPT are similar, and $\angle \text{H}t\text{P} = \angle \text{PST}$;

therefore $\angle t\text{OS} = \angle t\text{PS} = \angle \text{S}g\text{C}$.

Similarly, the triangles SPt, HPT are similar, and $\angle t\text{SP} = \angle \text{PTH}$;

therefore the triangles tPS, tO′T are similar, and $\angle t\text{O}'\text{T} = \angle t\text{PS}$;

therefore $\angle t\text{OS} = \angle t\text{O}'\text{T} = \angle \text{S}g\text{C}$;

therefore O, O′ lie on a circle whose centre is g and radius gS.

6087. (By C. Leudesdorf, M.A.)—The tangent at any point P on the curve $ay^2 = x^3$ cuts the curve again at Q; the tangent at Q cuts the curve again at R; and the tangent at R cuts PQ in S: shew that the locus of S, as P moves along the curve, is $343ay^2 = 100x^3$.

Solution by T. R. Terry, M.A.; E. W. Symons, B.A.; *and others.*

If the coordinates of P are (ap^2, ap^3), the equation to the tangent at P is
$$y-\tfrac{3}{2}px = -\tfrac{1}{2}ap^3 \quad\ldots\ldots\ldots\ldots (1).$$
Hence, to find abscissa of Q, we have $x^3-\tfrac{9}{4}ap^2x^2+\ldots = 0$;
therefore the coordinates of Q are $(\tfrac{1}{4}ap^2, -\tfrac{1}{8}ap^3)$. Similarly, the coordinates of R are $(\tfrac{1}{16}ap^2, \tfrac{1}{64}ap^3)$; therefore the equation to tangent at R is
$$y-\tfrac{3}{8}px = -\tfrac{1}{128}ap^3 \quad\ldots\ldots\ldots\ldots (2).$$
Eliminating p between (1) and (2), we get $343ay^2 = 100x^3$.

5877. (By Professor Townsend, F.R.S.)—Find the Potential, for the law of the inverse cube of the distance, of an elliptic lamina of uniform density, at any point in its space, as a function of the least semi-axis c of its confocal ellipsoid of equilibrium passing through the point.

Solution by the Proposer.

The attraction C of the lamina at the positive extremity of the least axis 2C of any ellipsoid of equilibrium of its mass (see *Reprint*, Vol. XXV., Quest. 4899) being $= -\dfrac{\mu fm}{abc}$, where m is the mass of the lamina and a, b, c the three semi-axes of the ellipsoid, or $= -\dfrac{\mu fm}{c(c^2+h^2)^{\frac{1}{2}}(c^2+k^2)^{\frac{1}{2}}}$, where h and k are the semi-axes of the lamina; and being also, at the same point, $= -\mu f.\dfrac{dV}{dc}$, where V is the required potential; therefore at once, for that point, and consequently for every point on the same ellipsoid of equilibrium abc, $\dfrac{dV}{dc} = \dfrac{m}{c(c^2+h^2)^{\frac{1}{2}}(c^2+k^2)^{\frac{1}{2}}}$; and therefore, by integration in finite terms, remembering that $V = 0$ when $c = \infty$,
$$V = -\tfrac{1}{2}\frac{m}{hk}\log\left\{\frac{\left(\dfrac{1}{h^2}+\dfrac{1}{k^2}+\dfrac{2}{c^2}\right)+2\left[\dfrac{1}{h^2k^2}+\left(\dfrac{1}{h^2}+\dfrac{1}{k^2}\right)\dfrac{1}{c^2}+\dfrac{1}{c^4}\right]^{\frac{1}{2}}}{\left(\dfrac{1}{h}+\dfrac{1}{k}\right)^2}\right\};$$
which accordingly is the value required.

When $h = k = r$, that is, for a circular lamina of radius r, the above value for V assumes the very simplified form $V = -\dfrac{m}{r^2}\log\dfrac{(r^2+c^2)^{\frac{1}{2}}}{c}$; which may be immediately verified directly for that simple case.

6082. (By F. MORLEY, B.A.)—If the straight line joining any point P on an hyperbola to the further vertex A′ meet the nearer directrix in Q; prove that PQ, QA′ subtend equal angles at the nearer focus S.

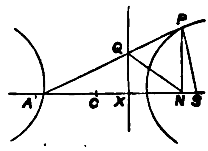
Solution by R. E. RILEY, B.A.; G. HEPPEL, M.A.; *and others.*

Let N be the foot of the ordinate of P, and X the foot of the directrix; then we have

$$\frac{PQ}{QA'} = \frac{NX}{A'X} = \frac{CN-CX}{CA+CX}$$

$$= \frac{e\,.\,CN-CA}{CS+CA} = \frac{SP}{SA'};$$

hence SQ bisects the angle A′SP.



6104. (By EDWYN ANTHONY, M.A.)—Prove that

$$(2a+b+c)\,(2b+c+a)\,(2c+a+b) > 8\,(a+b)\,(b+c)\,(c+a) \;\ldots\ldots\; (1);$$

$$a^2b^2+b^2c^2+c^2a^2 > abc\,(a+b+c) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Solution by R. E. RILEY, B.A.; D. EDWARDES; *and others.*

It is well known that

(i.) $(y+z)(z+x)(x+y) > 8xyz$; (ii.) $x^2+y^2+z^2 > yz+zx+xy$.

In (i.) for x write $b+c\ldots$, and in (ii.) for x write $bc\ldots$; therefore, &c.

5718. (By J. J. WALKER, M.A.)—Referring to Professor WOLSTENHOLME's Question 5692, show that, generally,

$$U_n = \int_0^{\frac{1}{4}\pi} (\log \tan x)^2 \; dx = 1\,.\,2 \ldots 2n \left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots\right);$$

and, to verify this general formula by comparison with the particular results given in Question 5692, show that it is equal to $A_{2n+1}\,(\frac{1}{4}\pi)^{2n+1}$, where A_{2n+1} is the value of $\dfrac{d^{2n+1}\log(1+\tan x)}{dx^{2n+1}}$ when $x=0$.

Solution by ROBERT RAWSON.

Put $\tan x = z$, therefore $dx = \dfrac{dz}{1+z^2}$; when $x=0$, then $z=0$; when $x=\frac{1}{4}\pi$, then $z=1$. Hence we have

$$U_n = \int_0^1 (\log z)^{2n} \frac{dz}{1+z^2} = \int_0^1 (\log x)^{2n} \left\{1 - x^2 + x^4 - x^6 + \ldots\right\} dx$$

$$= 1 \text{ to } 2n \left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots\right\} \;\ldots\ldots\ldots\ldots\; (1).$$

Equation (1) is easily obtained from the well-known formula of reduction as given by WILLIAMSON in his *Integral Calculus*, page 75, where it is shewn as follows:

$$\int (\log x)^{2n} x^{2m} dx = \frac{x^{2m+1}(\log x)^{2n}}{2m+1} - \frac{2n}{2m+1}\int (\log x)^{2n-1} x^{2m} dx,$$

therefore $$\int_0^1 (\log x)^{2n} x^{2m} dx = -\frac{2n}{2m+1}\int_0^1 (\log x)^{2n-1} x^{\ m} dx$$

$$= \frac{1 \text{ to } 2n}{(2m+1)^{2n}}\int_0^1 x^{2m} dx = \frac{1 \text{ to } 2n}{(2m+1)^{2n+1}} \ldots\ldots(2).$$

By giving to (m), in (2), the values 1, 2, 3, &c. in succession, the value of U_n, in (1), is obvious.

The very beautiful relation (1), which expresses the sum of

$$\left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots \text{ to infinity}\right)$$

by a means of a definite integral, is due to Professor WOLSTENHOLME.

Before verifying (1) by comparison with the particular results of Question 5692, the following observations seem to be necessary.

For the summation to infinity of the series in (1), reference might be made to LANDEN'S *Mathematical Lucubrations*, Part V., pages 37—46, and to SPENCE'S *Logarithmic Transcendents*, pages 51—70.

Series of the kind contained in (1) received considerable attention from the BERNOULLIS and EULER. JOHN BERNOULLI first resolved $\sin x = 0$ into its binomial factors,—a great step; but EULER was the first to sum to infinity the series in (1), by a process similar, if not the same, to the following:—I have used p to $q = p \,.\, p+1, p+2 \ldots q$ as being convenient to write and print.

The roots of $1 + \tan x = 0$ are $\frac{3}{4}\pi$, $\frac{7}{4}\pi$, $\frac{11}{4}\pi$, ... $\frac{1}{4}(4m-1)\pi$, and $-\frac{1}{4}\pi$, $-\frac{5}{4}\pi$, $-\frac{9}{4}\pi$, ... $-\frac{1}{4}(4m-3)\pi$.

It readily follows, from the well known doctrine of equations, that

$$\log(1+\tan x) = \log\left(1 + \frac{4x}{\pi}\right) + \log\left(1 - \frac{4x}{3\pi}\right)$$

$$+ \ldots \log\left(1 + \frac{4x}{(4m-3)\pi}\right) + \log\left(1 - \frac{4x}{(4m-1)\pi}\right)$$

$$= +\frac{4}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \text{\&c.}\right\}x - \frac{4^2}{2\pi^2}\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right\}x^2$$

$$+ \frac{4^3}{3\,.\,\pi^3}\left\{1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right\}x^3 - \frac{4^4}{4\pi^4}\left\{1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right\}x^4$$

$$+ \ldots - \frac{4^{2n}}{2n\pi^{2n}}\left\{1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \ldots\right\}x^{2n}$$

$$+ \frac{4^{2n+1}}{(2n+1)\,\pi^{2n+1}}\left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots\right\}x^{2n+1} \ldots\ldots\ldots\ldots\ldots (3).$$

By MACLAURIN'S theorem, the coefficient of x^{2n+1} in the development of $\log(1+\tan x)$ is $\frac{d^{2n+1}}{dx^{2n+1}}\log(1+\tan x)$, when $x = 0$, upon 1 to $(2n+1)$.

Hence, from (3), we obtain

$$U_n = 1 \text{ to } 2n \left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right)$$

$$= (\tfrac{1}{4}\pi)^{2n+1}\left(\frac{d^{2n+1}}{dx^{2n+1}}\log(1+\tan x) \text{ when } x = 0\right) \dots\dots\dots\dots\dots\dots\dots (4).$$

Equation (4) is the property enunciated in the question.

Since the roots of $\cos(\frac{1}{4}\pi + x) = 0$ are equal to the roots of $1 + \tan x = 0$ with the signs changed, it follows that

$$\frac{\cos x + \sin x}{\cos x - \sin x} = \frac{1}{\cos 2x} + \tan 2x = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

$$+ \frac{4^2}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)x + \dots + \frac{4^{2n}}{\pi^{2n}}\left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots\right)x^{2n-1}$$

$$+ \frac{4^{2n+1}}{\pi^{2n+1}}\left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right)x^{2n} \dots\dots\dots\dots\dots\dots\dots\dots (5).$$

Hence $1 \text{ to } 2n\left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right) = (\tfrac{1}{4}\pi)^{2n+1} \cdot \tfrac{1}{2}Q_{2n} = U_n \dots\dots (6)$,

where Q_{2n} upon $(1 \text{ to } 2n)$ is the coefficient of x^{2n} in $\sec x$.

Put $$\frac{1}{\cos x} = 1 + \frac{Q_2}{1 \text{ to } 2} \cdot x^2 + \frac{Q_4}{1 \text{ to } 4} \cdot x^4 + \dots \frac{Q_{2n}}{1 \text{ to } 2n} \cdot x^{2n},$$

therefore

$$1 = \left(1 - \frac{x^2}{1 \text{ to } 2} + \frac{x^4}{1 \text{ to } 4} - \dots\right)\left(1 + \frac{Q_2}{1 \text{ to } 2} \cdot x^2 + \frac{Q_4}{1 \text{ to } 4} \cdot x^4 + \dots\right).$$

From this equation the values of Q_2, &c. are readily calculated.

$$Q_2 = 1, \quad Q_4 = \frac{Q_2 \cdot 3 \cdot 4}{1 \cdot 2} - 1 = 5, \quad Q_6 = \frac{Q_4 \cdot 5 \cdot 6}{1 \cdot 2} - \frac{Q_2 \cdot 5 \cdot 6}{1 \cdot 2} + 1 = 61,$$

$$Q_8 = \frac{Q_6 \cdot 7 \cdot 8}{1 \cdot 2} - \frac{Q_4 \cdot 5 \text{ to } 8}{1 \text{ to } 4} + \frac{Q_2 \cdot 3 \text{ to } 8}{1 \text{ to } 6} - 1 = 1385,$$

$$Q_{10} = \frac{Q_8 \cdot 9 \cdot 10}{1 \cdot 2} - \frac{Q_6 \cdot 7 \text{ to } 10}{1 \text{ to } 4} + \frac{Q_4 \cdot 5 \text{ to } 10}{1 \text{ to } 6} - \frac{Q_2 \cdot 3 \text{ to } 10}{1 \text{ to } 8} + 1 = 50521,$$

$$Q_{12} = \frac{Q_{10} \cdot 11 \cdot 12}{1 \cdot 2} - \frac{Q_8 \cdot 9 \text{ to } 12}{1 \text{ to } 4} + \frac{Q_6 \cdot 7 \text{ to } 12}{1 \text{ to } 6} - \frac{Q_4 \cdot 5 \text{ to } 12}{1 \text{ to } 8} + \frac{Q_2 \cdot 3 \text{ to } 12}{1 \text{ to } 10} - 1 = 2{,}702{,}765,$$

$$Q_{14} = \frac{Q_{12} \cdot 13 \cdot 14}{1 \cdot 2} - \frac{Q_{10} \cdot 11 \text{ to } 14}{1 \text{ to } 4} + \frac{Q_8 \cdot 9 \text{ to } 14}{1 \text{ to } 6} - \frac{Q_6 \cdot 7 \text{ to } 14}{1 \text{ to } 8} + \frac{Q_4 \cdot 5 \text{ to } 14}{1 \text{ to } 10} - \frac{Q_2 \cdot 3 \text{ to } 14}{1 \text{ to } 12} + 1 = 199{,}360{,}981,$$

$$Q_{16} = \frac{Q_{14} \cdot 15 \cdot 16}{1 \cdot 2} - \frac{Q_{12} \cdot 13 \text{ to } 16}{1 \text{ to } 4} + \frac{Q_{10} \cdot 11 \text{ to } 16}{1 \text{ to } 6} - \frac{Q_8 \cdot 9 \text{ to } 16}{1 \text{ to } 8} + \frac{Q_6 \cdot 7 \text{ to } 16}{1 \text{ to } 10} - \frac{Q_4 \cdot 5 \text{ to } 16}{1 \text{ to } 12} + \frac{Q_2 \cdot 3 \text{ to } 16}{1 \text{ to } 14} - 1 = 19{,}391{,}512{,}145,$$

$Q_{18} = 2{,}404{,}879{,}675{,}441,$

$Q_{20} = 370{,}371{,}188{,}237{,}525.$

From the above values the definite integral $U_{10} = \int_0^{\frac{1}{4}\pi} (\log \tan x)^{20} dx$ is obtained. It is readily seen, from (2), that

$$1 \text{ to } 2n \left\{ 1 + \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} + \dots \right\} = \int_0^1 (\log x)^{2n} \frac{dx}{1-x^2} \dots\dots\dots (7).$$

The reason of the terminal digits 1, 5 occurring alternately, I apprehend, is to be found in the formula for the coefficients of x^2, x^4, &c., in the expansion of sec x.

Note on Criticoids and Synthetical Solution.

By Sir James Cockle, F.R.S.

Synthetical solution (*Reprint*, Vol. XXVI., pp. 59—61) may be extended to terordinals. Combined with the theory of criticoids (*Ibid.*, Vol. IX., pp. 105—112), it gives results. Call the equation formed by expunging the last term of a linear differential equation its *cæsura*. Consider a terordinal, wherein the coefficient of the last term is a constant. Differentiate it. If the criticoid of the *cæsura* (treated as a biordinal) of the derived be the same as that of the *cæsura* (similarly treated) of the original terordinal, both terordinals may be, and, in a particular case before me, actually are, depressible by one order. The depression is a consequence of the identity of the criticoids happening to render the middle operators of the synthetical solutions identical.

6065 & 6096. (By Prof. Crofton, F.R.S.)—

(6065) If $u_n = x^n + \frac{n(n-1)}{2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} + \text{\&c.}$,

prove that $x^n = u_n - \frac{n(n-1)}{2} u_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} - \text{\&c.}$

(6096.) If $u_n = v_n + a_n v_{n-2} + \frac{a_n a_{n-2}}{1 \cdot 2} v_{n-4} + \frac{a_n a_{n-2} a_{n-4}}{1 \cdot 2 \cdot 3} v_{n-6} + \text{\&c.}$,

prove that $$v_n = u_n - a_n u_{n-2} + \frac{a_n a_{n-2}}{1 \cdot 2} u_{n-4} - \text{\&c.}$$

Solution by the Rev. J. L. Kitchin, M.A.; T. R. Terry, M.A.; *and others.*

(6065.) Let $E = u_n - \frac{n(n-1)}{2} u_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} u_{n-4} - \dots$,

then the coefficient of x^{n-2r} in E is

$$\frac{n(n-1)\dots(n-2r+1)}{2^r\,.\,r\,!} - \frac{n(n-1)}{1\,.\,2}\cdot\frac{(n-2)\dots(n-2r+1)}{2^{r-1}(r-1)\,!}$$
$$+\frac{n(n-1)(n-2)(n-3)}{2^2\,.\,2\,!}\cdot\frac{(n-4)\dots(n-2r+1)}{2^{r-2}(r-2)\,!} - \&c.$$
$$= \frac{n(n-1)\dots(n-2r+1)}{2^r\,.\,r\,!}(1-1)^r, \text{ therefore } \mathrm{E} = x^n.$$

(6069.) Let $\quad \mathrm{E} = u_n - a_n u_{n-2} + \dfrac{a_n a_{n-2}}{1\,.\,2} u_{n-4} - \&c.,$

then the coefficient of v_{n-2r} in E is

$$\frac{a_n \dots a_{n-2r+2}}{r\,!} - a_n \frac{a_{n-2}\dots a_{n-2r+2}}{(r-1)\,!} + \frac{a_n a_{n-2}}{1\,.\,2}\,\frac{a_{n-4}\dots a_{n-2r+2}}{(r-2)\,!} - \&c.$$
$$= \frac{a_n \dots a_{n-2r+2}}{r\,!}\left\{1 - r + \frac{r(r-1)}{1\,.\,2} - \dots\right\} = \frac{a_n\dots a_{n-2r+2}}{r\,!}(1-1)^r,$$

therefore $\quad \mathrm{E} = v_n.$

[Prof. Crofton's solution of both Questions is very simple. Let ζ be an operator defined by $\quad \zeta f(n) = \dfrac{n(n-1)}{2}\,.\,f(n-2),$

then we have $\quad u_n = x^n + \zeta x^n + \dfrac{\zeta^2}{1\,.\,2} x^n + \&c = e^{\zeta} x^n,$

therefore $\quad x^n = e^{-\zeta} u_n = u_n - \dfrac{n(n-1)}{2} u_{n-2} + \&c.$

In like manner, we may prove the property stated in Quest. 6096.]

6054. (By W. H. H. Hudson, M.A.)—Trace the curves

$$(y-x)^2 + 2(y-x)^2(y+x) + x^4 + y^4 = 0 \;\dots\dots\dots\dots\dots\dots (1),$$
$$(y-x)^2 + 2(y-x)(y^2+x^2) + x^4 + y^4 = 0 \;\dots\dots\dots\dots\dots\dots (2).$$

Solution by Professor Wolstenholme, M.A.

(1). Here we have $(y-x)^2(1+x+y)^2 = -2x^2y^2$; hence, the only real points are

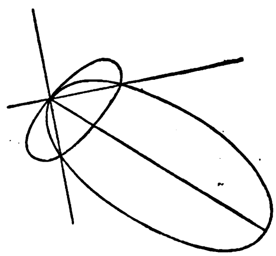
$(x=0,\ y=0),\quad (x=0,\ y=-1),$
$(y=0,\ x=-1).$

(2). Here $(y-x+y^2+x^2)^2 = 2x^2y^2$

or $\quad x^2 \pm \sqrt{2}xy + y^2 + y - x = 0,$

two similar ellipses touching each other and the line $y = x$ at the origin, which is the end of the major axis of one and the minor axis of the other. The axes of either are as

$\sqrt{2}+1 : 1.$



5965. (By the EDITOR.)—Find the nature of the roots of the equation

$$e^{x^x+x\log x}+14e^{-x^x-x\log x}=8,$$

and show that $x = \cdot 0047$.

Solution by Professor COCHEZ.

Posons $e^{x^x+x\log x}=z$; l'équation devient

$$z+14z^{-1}=8;\ \text{d'où}\ z^2-8z+14=0,\ \text{et}\ z=4\pm\sqrt{2};$$

par suite $\qquad z_1=4+\sqrt{2},\quad z_2=4-\sqrt{2}.$

Prenons d'abord z_1. Nous avons:

$$e^{x^x+x\log x}=4+\sqrt{2}$$

et, si nous prenons les logarithmes népériens, $x^x+x\log x=\log(4+\sqrt{2})$. Posons $x^x=t$; donc $x\log x=\log t$, $t+\log t=\log(te^t)=\log(4+\sqrt{2})$, d'où $te^t=4+\sqrt{2}$. Pour $t=1$ le premier membre est égal $2{\cdot}71$; pour $t=2$ il est supérieur à $4+\sqrt{2}$. Cette équation admet donc une racine comprise entre 1 et 2.

D'ailleurs te^t croit constamment quand on fait varier t depuis 0 jusqu'à $+\infty$. Donc l'équation $te^t=4+\sqrt{2}$ n'a qu'une seule racine, et cette racine est comprise entre 1 et 2.

Soit $1+\alpha$ sa valeur, on a $x^x=1+\alpha$. Cherchons combien cette équation a de racines. Le premier membre doit être positif, par suite x doit être positif. Etudions la marche de la fonction x^x lorsqu'on fait varier x depuis 0 jusqu'à $+\infty$.

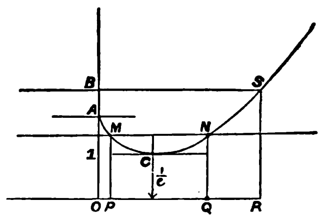
$y=x^x$, dont la dérivée est $y'=x^x(1+\log x)$. Quand x est très-petit, $\log x$ est très-grand et négatif; pour x très-petit la parenthèse est négative, la dérivée est négative, et par suite la fonction est décroissante.

Pour $x=0$, $\qquad \log y=0$, d'où $y=1$, $y'=-\infty$,

et y décroîtra tant qu'on aura

$1+\log x<0$, $\log x<-1$, ou $<e^{-1}$;

lorsque x est négatif la fonction est discontinue, et il n'y a plus de courbe.



Revenons à l'équation $x^x=1+\alpha$. Cette équation n'a qu'une racine. Puisque cette racine correspond à une valeur de $y>1$, c'est-à-dire plus grande que OA. Toute parallèle distante de l'axe des x d'une quantité plus grande que 1 ne rencontre la courbe qu'en un point. Il n'y a donc qu'une racine correspondant à la valeur $z_1=4+\sqrt{2}$.

Considérons maintenant la valeur $z_2=4-\sqrt{2}=te^t$ ....(1).

Cette équation a une racine réele comprise entre 0 et 1, et n'en a qu'une. Pour $t=0$ le 1er nombre est nul, pour $t=1$ le 1er membre est égal à $2{\cdot}71$; et, comme le second membre est égal environ à $2{\cdot}6$, il en résulte qu'il y a une racine comprise entre 0 et 1.

Soit $(1-\beta)$ cette racine, on a alors $x^x=1-\beta$. La racine de l'équation en t est très-voisine de 1, par conséquent β est très-petit. Donc la parallèle à l'axe des x, qui par son intersection avec la courbe donnera les racines de la dernière équation, tombe entre les points A et C, et nous voyons

qu'il y a deux racines, l'une comprise entre 0 et e^{-1} et l'autre comprise entre e^{-1} et 1. L'équation proposée a donc en tout 3 racines, et la plus petite est celle qui est comprise entre 0 et e^{-1}.

On calcule la racine (1) par la méthode des substitutions successives, et on trouve $t = 0{\cdot}9752$.

On résout alors l'équation $x^x = t = 0{\cdot}9752$, et on trouve $x = 0{\cdot}0047$.

5667. (By Christine Ladd.)—Express $\tan 2A + \tan 2B + \tan 2C$ and $\cot 2A + \cot 2B + \cot 2C$ in terms of the radii of circles connected with a given triangle ABC.

Solution by the Proposer.

The angles of the orthocentric triangle $P_1P_2P_3$ are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$, and the radius of its circumscribed circle is $\frac{1}{2}R$; therefore the radius of the circle inscribed in the triangle orthocentric to $P_1P_2P_3$ is

$$\rho' = -R \cos 2A \cos 2B \cos 2C.$$

The area of the triangle $P_1P_2P_3$ is $rs\rho R^{-1}$. The sum of the tangents of the triangle ABC is $\frac{\Delta ABC}{R\rho}$; and, similarly,

$$\tan 2A + \tan 2B + \tan 2C = \frac{2rs\rho}{R^2\rho'} = \frac{2\rho}{R^2\rho'}(rr_1r_2r_3)^{\frac{1}{2}}.$$

Since $\sin A \sin B \sin C = \frac{1}{2}rsR^{-2}$, we have, for the triangle $P_1P_2P_3$,

$$\sin 2A \sin 2B \sin 2C = \frac{2rs\rho}{R^3};$$

$$\cot A + \cot B + \cot C = \frac{\cos A \cos B \cos C + 1}{\sin A \sin B \sin C} = \frac{R(\rho + 2R)}{r^3}.$$

Similarly
$$\cot 2A + \cot 2B + \cot 2C = \frac{-\cos 2A \cos 2B \cos 2C + 1}{\sin 2A \sin 2B \sin 2C}$$
$$= \frac{R^2(\rho' + R)}{2rs\rho} = \frac{R^2(\rho' + R)}{2\rho(rr_1r_2r_3)^{\frac{1}{2}}}.$$

5940. (By R. Tucker, M.A.)—Cut a straight line into three parts, so that the rectangle under the whole line and the mid-section (of given length) shall be equal to the rectangle under the outside segments; and find the limits of possibility.

Solution by W. J. MACDONALD, M.A.; J. O'REGAN; *and others.*

A K L B C D G E

Let AB be the given line, and C the given mid-section. Take DE = AB − C. Divide DE so that the rectangle DG . GE = rectangle AB . C; cut off AK = DG, KL = C; therefore LB = GE, and rectangle AB . KL = AK . LB. The rectangle AB . C must not be > $\frac{1}{4}$ sq. on (AB − C).

6081. (By Dr. MACFARLANE, F.R.S.E.)—In a certain collection of objects, those having a quality X, together with those having a quality Y, together with those having a quality Z, are identical with those having a quality A, together with those having a quality B. The qualities X, Y, Z are not necessarily mutually exclusive, nor the qualities A and B. What conclusions may be immediately deduced?

Solution by the PROPOSER.

The given relation is $\quad x+y+z = a+b.$

Now $\quad x+y+z = 3xyz + 2xy(1-z) + 2yz(1-x) + 2zx(1-y)$

$$+x(1-y)(1-z) + y(1-z)(1-x) + z(1-x)(1-y),$$

and $\quad a+b = 2ab + a(1-b) + b(1-a)$;

therefore $\quad 3xyz + 2xy(1-z) + 2yz(1-x) + 2zx(1-y)$

$$+x(1-y)(1-z) + y(1-z)(1-x) + z(1-x)(1-y)$$
$$= 2ab + a(1-b) + b(1-a).$$

Now, since the terms of either member are mutually exclusive, the part which is taken three times on the one side must be identical with the part which is taken three times on the other side, and so on;

therefore $\quad xyz = 0, \quad xy(1-z) + yz(1-x) + zx(1-y) = ab$ ....(1, 2),

$x(1-y)(1-z) + y(1-z)(1-x) + z(1-x)(1-y) = a(1-b) + b(1-a)$ ...(3).

By combining (1) and (2), we have $xy + yz + zx = ab$.

6028. (By T. MORLEY, L.C.P.)—Solve the equations

$$(x+2y)(x+2z) = a^2, \quad (y+2z)(y+2x) = b^2, \quad (z+2x)(z+2y) = c^2.$$

Solution by Professor GOLDENBERG, Moscow.

En posant $\quad x+y+z = t, \quad xy+yz+zx = u,$

j'additionne membre à membre les équations données, et j'obtiens une première relation entre t et u, $\quad t^2 + 6u = a^2 + b^2 + c^2$ ....(1).

Multipliant membre à membre la première des equations données par la seconde, puis par la troisième, puis la seconde par la troisième, et faisant la somme des produits obtenus, j'aurai

$$21(x^2+y^2+z^2)+48(x^2yz+xy^2z+xyz^2)+6(x^3y+xy^3+y^3z+yz^3+z^3x+zx^3) = a^2b^2+b^2c^2+c^2a^2,$$

ce qui, après quelques reductions, me donne une seconde relation entre t et u, savoir $\quad 3u^2+2tu = \frac{1}{3}(a^2b^2+b^2c^2+c^2a^2)$ ....(2).

Eliminant t entre (1) et (2), j'obtiens une équation du second dégré en u,

$$9u^2-2(a^2+b^2+c^2)u+\tfrac{1}{3}(a^2b^2+b^2c^2+c^2a^2)=0;$$

d'où, en posant $\quad (a^4+b^4+c^4)-(b^2c^2+c^2a^2+a^2b^2)=k^4,$

j'aurai $\quad u=\frac{1}{9}(a^2+b^2+c^2\pm k^2)$, et $t^2=\frac{1}{3}(a^2+b^2+c^2\mp 2k^2)$.

Pour avoir maintenant x, par exemple, je remplace, dans l'équation $x^2+2x(y+z)+4yz=a^2$, $y+z$ par $t-x$, yz par $u-x(t-x)$, et je trouve

$$3x^2-2tx+4u-a^2=0, \quad \text{d'où} \quad x=\tfrac{1}{3}\left\{t\pm[t^2-3(4u-a^2)]^{\frac{1}{2}}\right\}.$$

6038. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\begin{vmatrix} h^2+a^2-a\lambda, & h(a+b-\lambda), & g(a-\lambda)+fh \\ h(a+b-\lambda), & h^2+b^2-b\lambda, & f(b-\lambda)+gh \\ g(a-\lambda)+fh, & f(b-\lambda)+gh, & f^2+g^2-c\lambda \end{vmatrix} = -\Delta\lambda\left[(\lambda-a)(\lambda-b)-h^2\right],$$

where $\quad \Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$

Solution by T. R. TERRY, M.A.; G. HEPPEL, M.A.; *and others.*

$$\text{Determinant} = \left[(\lambda-a)(\lambda-b)-h^2\right]\begin{vmatrix} a, & h, & ga+fh-\lambda g \\ h, & b, & gh+fb-\lambda f \\ g, & f, & g^2+f^2-\lambda c \end{vmatrix}$$

$$= \left[(\lambda-a)(\lambda-b)-h^2\right]\begin{vmatrix} a, & h, & -\lambda g \\ h, & b, & -\lambda f \\ g, & f, & -\lambda c \end{vmatrix} = \text{\&c.}$$

6070. (By Professor MOREL.)—On donne un angle IAD et une droite mobile autour d'un point O. Trouver le lieu géométrique des centres des cercles circonscrits aux triangles successifs.

Solution by G. TURRIFF; G. HEPPEL, M.A.; *and others.*

Let A be the origin; AD ($=a$) the axis of x; (h, k) the point O; (c, mc) where m is a constant, the point I; then, because O is on ID, $\dfrac{a-c}{a-h} = \dfrac{mc}{k}$. Then, if (x, y) be the centre of the circumscribed circle, we have

$$x^2+y^2 = (x-a)^2+y^2 = (x-c)^2+(y-mc)^2.$$

Hence, eliminating a and c,

$$2x^2+2mxy-(mk+h)\,x-(mh-k)\,y = 0,$$

which is the equation to an hyperbola, and $x = \dfrac{mh-k}{2m}$ is an asymptote.

[This result suggests the following construction for the locus:—

Through O draw parallels to IA and DA, cutting those lines in E and F; and bisect AE and AF perpendicularly; then these perpendiculars are the asymptotes, and, since A is a point on the curve, the hyperbola is completely determined.]

6076. (By W. A. WHITWORTH, M.A.)—In a leap year selected at random prove that the chance that February 29 falls on Monday is greater than its chance of falling on Sunday in the ratio of 15 to 13.

Solution by G. HEPPEL, M.A.; J. A. KEALY, M.A.; *and others.*

Remembering that 2000 and 2400 are leap-years, but that 2100, 2200, 2300 are not, it is easy to form the following table showing the day of the week on which February 29th falls:—

2000 Tu.	2096 W.	2196 M.	2296 Sa.	2396 Th.
2088 Su.	2188 F.	2288 W.	2388 M.	2400 Tu.
2092 F.	2192 W.	2292 M.	2392 Sa.	

Hence, every year which is a multiple of 400 has February 29 on a Tuesday, and the same set of days recur. The leap-years omitted in the above table form four cycles of 84 years, in each of which February 29 occurs three times on every day of the week. Hence, in the whole 400 years, Sunday, Tuesday, and Thursday appear 13 times; Friday and Saturday 14 times; and Monday and Wednesday 15 times.

5979. (W. J. C. SHARP, M.A.)—Prove that the evectant of the Hessian of any curve is the reciprocal of the polar conic of (x, y, z) multiplied by a suitable power of $\alpha x+\beta y+\gamma z$, and the second evectant vanishes identically.

Solution by J. J. WALKER, M.A.

The evectant of the Hessian of U is ($\mathrm{U} = a_0 x^n + \dots$)

$$\left(a^n\frac{d}{da_0}+\dots\right)\left(\frac{d}{dx_1},\ \frac{d}{dy_2},\ \frac{d}{dz_3}\right)\mathrm{U}_1\mathrm{U}_2\mathrm{U}_3,$$

the suffixes being dropped after [illegible]

[illegible]

or [illegible]

or [illegible]

But [illegible] for the second evolute,

[illegible]

which verifies identically.

The analogous properties [illegible] actual calculation in particular cases [illegible]

[illegible]

which is readily identified with

[illegible]

or [illegible]

5982. [illegible]

Solution by [illegible] *and others.*

The polar of [illegible] by comparing coefficients, [illegible] ordinate is easily found to be [illegible] thus the equation to normal [illegible] in the Question.

6010. (By Professor Townsend, F.R.S.)—A hollow truncated cone of revolution, filled with water and placed with axis vertical on a horizontal plane, being supposed pierced all over its conical area with small circular apertures through which the water issues perpendicularly with the velocity

due to the depth below its free surface; determine, given all particulars, the envelope of the several jets, and from it the position of the circle of issue on the zone corresponding to that of the maximum range of the water on the plane.

Solution by the PROPOSER.

The several jets along any generating line of the zone being manifestly, under the circumstances supposed, a system of similar and similarly placed parabolas, having a common centre of similitude at the upper extremity of the line, a common horizontal directrix passing through the same extremity, and a common normal direction along the entire extent of the line, their envelope is consequently a right line, passing through the aforesaid upper extremity of the generating line, and bisecting the angle between the common horizontal directrix and the line of foci of all the parabolas. Hence, for the entire zone, the envelope of the entire system of jets is another truncated cone of revolution coaxal with the zone, starting from its upper circle, and intersecting the horizontal plane of the lower in a concentric circle, which is manifestly that of the maximum range of the water on the plane, and which as manifestly determines at once, when all other particulars are given, the corresponding circle of issue of the water on the surface of the zone.

6103. (By C. LEUDESDORF, M.A.)—TP, TQ are tangents to an hyperbola. The circle passing through T, P, Q cuts the hyperbola again in P′, Q′, tangents at which to the hyperbola intersect in T′. Show that CT . CT′ = CS², C being the centre, and S a focus, of the hyperbola.

Solution by Professor WOLSTENHOLME, M.A.

If from two points T, T′ tangents TP, TQ, T′P′, T′Q′ be drawn to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the six points TPQ, T′P′Q′ all lie (x_1y_1, x_2y_2 being the coordinates of T, T′) on the one conic

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right).$$

This will be a circle if $x_1y_2+x_2y_1=0$ and $x_1x_2-y_1y_2=a^2-b^2$; giving, geometrically interpreted, CT, CT′ equally inclined to the axis, and CT . CT′ = CS². [A parabola drawn with focus T′ and touching the axes will touch the polar of T and the normals at its ends. See Prof. WOLSTENHOLME's *Book of Mathematical Problems*, 2nd ed., Quest. 1207.]

5902. (By Professor TOWNSEND, F.R.S.)—An annular lamina of uniform density, bounded externally and internally by concentric circles of any radii, being supposed to attract, according to the law of the inverse cube of the distance, any element of its mass; show that its attraction is

the suffixes being dropped after operating; which is

$$\left(\frac{d}{dx_1}, \dots\right)^2 \left\{(\alpha x_1 + \beta y_1 + \gamma z_1)^n \, U_2 U_3 + \dots\right\},$$

or
$$\left(\frac{d}{dx_1}, \dots\right)^2 (\alpha x_1 + \dots)^n \, U_2 U_3,$$

or
$$n(n-1)(\alpha x + \dots)^{n-2} \left(\alpha, \frac{d}{dy_2}, \frac{d}{dz_3}\right)^2 U_2 U_3.$$

But $(\alpha 12)^2 \, U = (bc - f^2)\,\alpha^2 + \dots$; and, by a similar process, we should find, for the second evectant,

$$n^2 (n-1)^2 (\alpha x + \dots)^{2n-4} \left(\alpha, \beta, \frac{d}{dz_3}\right) U_3,$$

which vanishes identically.

The analogous properties of binary quantics may be easily verified by actual calculation in particular cases; *e.g.*, the first evectant of the Hessian $(abcde)(xy)^4$ is

$$(cx^2 + 2dxy + ey^2)\,\alpha^4 - 2\,(bx^3 + cx^2y - dxy^2 - ey^3)\,\alpha^3\beta$$
$$+ (ax^4 - 2bx^3y - 6cx^2y^2 - 2dxy^3 + ey^4)\,\alpha^2\beta^2 - \dots,$$

which is readily identified with

$$(\alpha x + \beta y)^2 \left\{(cx^2 + 2dxy + ey^2)\alpha^2 - 2(bx^2 + 2cxy + dy^2)\,\alpha\beta + (ax^2 + 2bxy + cy^2)\,\beta^2\right\}$$

or
$$4\,.\,3\,(\alpha x + \beta y)^2 \left(\alpha^2 \frac{d^2u}{dy^2} - 2\alpha\beta \frac{d^2u}{dxdy} + \beta^2 \frac{d^2u}{dx^2}\right).$$

5993. (By J. J. Walker, M.A.)—If PQ is a chord of the parabola $y^2 = 4ax$, normal at P, and $(\xi\eta)$ is its pole, prove that the equation to the normal at Q is $8a^5y + \xi\eta\,(-4a^3x + 8a^4 + \xi^2\eta^2) = 0$, the corresponding form to which for central conics is given in Quest. 5095 (*Reprint*, Vol. XXVI., p. 99).

Solution by R. Knowles, B.A., L.C.P.; R. E. Riley, B.A.; *and others.*

The polar of (ξ, η) is $\eta y = 2a\,(x + \xi)$; if this be normal at P (h, k), we get, by comparing coefficients, $8a^2 + k^2 = -4a\xi$, $k = -4a^2\eta^{-1}$; also at Q the ordinate is easily found to be $-\left(\dfrac{8a^2 + k^2}{k}\right) = -\dfrac{\xi\eta}{a}$; thus the equation to normal at Q is $y = -\dfrac{\xi\eta}{a} + \dfrac{\xi\eta}{2a^2}\,x - \dfrac{\xi^3\eta^3}{8a^5}$, or &c., as in the Question.

6010. (By Professor Townsend, F.R.S.)—A hollow truncated cone of revolution, filled with water and placed with axis vertical on a horizontal plane, being supposed pierced all over its conical zone with small circular apertures through which the water issues perpendicularly with the velocity

5372. (By Professor MINCHIN, M.A.)—Prove that the equation of the Catenary of Uniform Strength, expressed in terms of the radius of curvature, and the arc measured from the lowest point, is

$$\rho = \tfrac{1}{2}a\left(e^{\frac{s}{a}} + e^{-\frac{s}{a}}\right).$$

Solution by J. HAMMOND, M.A.; W. J. MACDONALD, M.A.; *and others.*

In the Catenary of Uniform Strength, the weight of an element of the string is proportional to the tension acting along the element, which put $= \dfrac{\mathrm{T}ds}{a}$; then we have $d(\mathrm{T}\sin\phi) = \dfrac{\mathrm{T}ds}{a}$, $\mathrm{T}\cos\phi = \text{const.}$

Substituting for T, we have

$$d(\tan\phi) = \frac{\sec\phi\, ds}{a} \quad\text{or}\quad \sec\phi\, d\phi = \frac{ds}{a} \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence, we have $$\frac{s}{a} = \log\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right),$$

$$e^{\frac{s}{a}} = \frac{1-\tan\frac{1}{2}\phi}{1+\tan\frac{1}{2}\phi},\qquad e^{-\frac{s}{a}} = \frac{1+\tan\frac{1}{2}\phi}{1-\tan\frac{1}{2}\phi},$$

$$e^{\frac{s}{a}} + e^{-\frac{s}{a}} = \frac{2\sec^2\frac{1}{2}\phi}{1-\tan^2\frac{1}{2}\phi} = \frac{2}{\cos\phi}.$$

From (1), since $\dfrac{ds}{d\phi} = \rho$, $\sec\phi = \dfrac{\rho}{a}$; therefore $\rho = \dfrac{a}{2}\left(e^{\frac{s}{a}} + e^{-\frac{s}{a}}\right)$.

5026. (By R. TUCKER, M.A.)—With the same data as in Question 5008 (*Reprint*, Vol. XXVII., p. 51),—R, R′, &c. being points on the axis through which chords are drawn at right angles to PQ, P′Q′, &c.,—show that (1) the common chords of the pairs of auxiliary circles pass through the vertex; (2) the lines joining mid-points of pairs of chords pass through the points on the axis where the normals corresponding to the abscissæ AR, AR′, &c., cut the axis; and (3) find the equations to the second common tangents.

Solution by the PROPOSER.

As in my solution of Question 5008, we find the equations to the auxiliary circles to be

$$x^2 + y^2 - 2\mathrm{X}\left(h + 2am^{-2}\right) - 4am^{-1}\,\mathrm{Y} = a^2 + 2ah \;\ldots\ldots\ldots\ldots\ldots(\text{i.}),$$

$$x^2 + y^2 - 2\mathrm{X}\left(h + 2am^2\right) + 4am\mathrm{Y} = a^2 + 2ah \;\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}).$$

Hence, equation to common chord is $\mathrm{Y} = (m - m^{-1})\,\mathrm{X}$, which proves (1). The equation to the line joining the mid-points of the chords is

$$y - 2am^{-1} = -\left(m - m^{-1}\right)^{-1}\left(x - h - 2am^{-2}\right) \;\ldots\ldots\ldots\ldots\ldots(\text{iii}),$$

which cuts the axis in $x = 2a + h$, and proves (2).

The line (iii.), which is also line of centres, cuts directrix in

$$y = (3a+h)(m-m^{-1});$$

therefore angle between line of centres and the directrix is given by

$$\tan\phi = m-m^{-1}, \text{ therefore } \tan 2\phi = \frac{2(m-m^{-1})}{1-(m-m^{-1})^2};$$

therefore the equation to the second common tangent is

$$y - \frac{3a+h}{m-m^{-1}} = \frac{m^2-1+m^{-2}}{2(m-m^{-1})}(x+a).$$

Hence all these tangents are parallel.

5871. (By W. H. H. Hudson, M.A.)—A solid hemisphere just fits into a thin hemispherical bowl of equal weight; show that, if r_1, r_2, r_3 are the radii of the spheres on the top of which (1) the bowl alone, (2) the solid alone, (3) the solid in the bowl, will rest in neutral equilibrium, then

$$r_1 : r_2 : r_3 = 21 : 35 : 27.$$

Solution by D. Edwardes; R. Tucker, M.A.; *and others.*

Let a be the radius of the bowl, and r_1, r_2, r_3 the radii of the spheres in (1), (2), (3) cases respectively; then (Minchin's *Statics*, p. 312), we have

(1) For neutral equilibrium, $\frac{1}{2}a = \frac{ar_1}{a+r_1}$, therefore $r_1 = a$.

(2) $\frac{5}{8}a = \frac{ar_2}{a+r_2}$, therefore $r_2 = \frac{5}{3}a$.

(3) Since the masses of the bowl and solid are equal, the height of the centre of inertia is $h = \frac{1}{2}a(\frac{1}{2}+\frac{5}{8}) = \frac{9a}{16}$, whence $\frac{9a}{16} = \frac{ar_3}{a+r_3}$, or $r_3 = \frac{9}{7}a$.
Hence, we have $r_1 : r_2 : r_3 = 21 : 35 : 27$.

5920. (By W. J. C. Sharp, M.A.)—Prove that no quartic can have three points of inflexion lying in the same straight line.

Solution by J. J. Walker, M.A.

The statement in this Question may be thus generalised: "If a curve of order n have $n-1$ points of inflexion in a right line, then here will be an n^{th} point of inflexion in the same line."

Let l be the line, then the equation to the curve $u = 0$ may be thrown into the form $pab \dots k = (p-1)l^2v$, where $ab \dots k$ are n linear forms and v is of the order $n-2$ (it being supposed that the leading term in all, as

well as in u, has unity for coefficient), since there are $1+2n+\frac{1}{2}(n\,.\,n-1)-1$ constants to be determined, and this number is exactly equal to that of the given constants in u, viz., $\frac{1}{2}(n+1\,.\,n+2)-1$. But the form plainly shows that $a, b \ldots k$ are the n tangents at the points where l meets in u; and if $n-1$ of these are points of inflexion, then l and v must have so many points in common, *i.e.*, v must be of the form lw, and the transformed equation is $pab \ldots k = (p-1)\, l^3 w$, so that the n^{th} point in which b meets u is also a point of inflexion. It may be observed that it has been shown that, *if n points of inflexion on u are collinear, then the $n\,(n-3)$ points in which the tangents at these points again meet u lie on a curve of order $n-3$*, which, in fact, is only a case of Gergonne's Theorem (SALMON's *Higher Plane Curves*, § 31).

3191. (By WALTER SIVERLY.)—Determine the maximum rectangle that can be inscribed between the cycloid and its base.

Solution by A. MARTIN, M.A.; *the* PROPOSER; *and others.*

Let (x, y) be the co-ordinates of a corner of the rectangle on the curve, then the base of the rectangle is $2\pi r-2x$, and the altitude is y; hence, we must have $(\pi r-x)\,y = \left\{\pi r-\left[\text{versin}^{-1} y-(2ry-y^2)^{\frac{1}{2}}\right]\right\} y = \max.$

$$\text{therefore}\quad \frac{du}{dy} = \pi r-\left[\text{versin}^{-1} y-(2ry-y^2)^{\frac{1}{2}}\right]\Big\} \mp \frac{y^2}{(2ry-y^2)^{\frac{1}{2}}} = 0,$$

$$\text{or}\qquad \frac{y^2}{(2ry-y^2)^{\frac{1}{2}}}-(2ry-y^2)^{\frac{1}{2}} = \pi r-\text{versin}^{-1} y.$$

Making $r=1$, and solving by approximation, we find $y = 1{\cdot}38866$, whence the base of the rectangle is readily found to be $4{\cdot}1856$.

5521. (By Professor SYLVESTER, F.R.S.) — If there be n quantities $a, b, \ldots k, l$, and if $(b, c, \ldots k, l)$ denote the product of the $\frac{1}{2}(n-1)(n-2)$ values of $1-pq$, when for p we put any one of the letters $b, c, k \ldots, l$, and for q any other of these letters; prove (1) that

$$\Sigma a^{n-2}\,\frac{(b, c, \ldots k, l)}{(a-b)(a-c)\ldots(a-l)} = 0\,;$$

and (2), more generally, that for all integer values of j not exceeding $n-2$,

$$\Sigma\,(a^{n-2+j}+a^{n-2-j})\,\frac{(b, c, \ldots k, l)}{(a-b)\,(a-c)\,\ldots\,(a-l)} = 0\,;$$

also (3), show that $(a, b, c, \ldots k, l)$ may be represented by a symmetrical determinant of the order $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd, of which $\Sigma a^{n-1}\,\dfrac{(b, c, \ldots k, l)}{(a-b)\,(a-c)\,\ldots\,(a-l)}$ is a symmetrical first minor.

Solution by W. J. CURRAN SHARP, M.A.

Let $(x-a)(x-b)\dots(x-l)\equiv f(x)$; then we have

$$.\,a^{n-2}\frac{(b\,.\,c\dots k\,.\,l)}{(a-b)(a-c)\dots(a-l)}=\Sigma\,.\,a^{n-2}\frac{(b\,.\,c\dots k\,.\,l)}{f'(a)}$$

$$=\Sigma\,.\,a^{n-2}(a\,.\,b\dots k\,.\,l)+\left\{(1-ab)(1-ac)\dots(1-al)f'(a)\right\}$$

$$=a\,.\,b\dots k\,.\,l)\,\Sigma\,\frac{a^{n-2}}{(1-ab)(1-ac)\dots(1-al)}\cdot\frac{1}{f'(a)}.$$

Now $$\frac{a^{n-2}}{(1-ab)(1-ac)\dots(1-al)}=\frac{b-a}{f'(b)}\cdot\frac{1}{1-ab}+\frac{c-a}{f'(c)}\cdot\frac{1}{1-ac}+\dots$$

$$\dots+\frac{l-a}{f'(l)}\cdot\frac{1}{1-al};$$

hence we have

$$\frac{a^{n-2}}{(1-ab)\dots(1-al)}\cdot\frac{1}{f'(a)}=\frac{b-a}{f'(b)f'(a)}\cdot\frac{1}{1-ab}+\frac{c-a}{f'(c)f'(a)}\cdot\frac{1}{1-ac}+\text{\&c.};$$

therefore $\Sigma\,\dfrac{a^{n-2}}{(1-ab)(1-ac)\dots(1-al)}\cdot\dfrac{1}{f'(a)}=0$, and equation (1) holds.

Again, $$\frac{a^j}{(1-ab)(a-ac)\dots(1-al)}=\frac{b^{n-2-j}(b-a)}{f'(b)}+\frac{c^{n-2-j}(c-a)}{f'(c)}+\text{\&c.};$$

$$\therefore\ \frac{a^{n-2+j}}{(1-ab)\dots(1-al)}\cdot\frac{1}{f'(a)}=\frac{a^{n-2}b^{n-2-j}(b-a)}{f'(a)f'(b)}+\frac{a^{n-2}c^{n-2-j}(c-a)}{f'(a)f'(c)}+\text{\&c.},$$

and

$$\frac{a^{n-2-j}}{(1-ab)\dots(1-al)}\cdot\frac{1}{f'(a)}=\frac{a^{n-2-j}b^{n-2}(b-a)}{f'(a)f'(b)}+\frac{a^{n-2-j}c^{n-2}(c-a)}{f'(a)f'(c)}+\text{\&c.};$$

therefore $\Sigma\,\dfrac{a^{n-2+j}+a^{n-2-j}}{(1-ab)(1-ac)\dots(1-al)}\cdot\dfrac{1}{f'(a)}=0\equiv P$; hence

$$(a,b,\dots k,l)\,.\,P\equiv\Sigma\,\frac{(b,c,\dots k,l)}{(a-b)(a-c)\dots(a-l)}(a^{n-2+j}a^{n-2-j})=0,$$

which proves equation (2).

6122. (By Professor CROFTON, F.R.S.)—The base AB of a triangle is bisected in D; if two points X, Y are taken, one in the triangle ADC, and the other in the triangle DCB, show that the chance that XY produced meets the base AB is $8\log 2-5$, or $\cdot 545$.

I. *Solution by* C. J. MONRO, M.A.; G. HEPPEL, M.A.; *and others.*

It is presumably intended that a point "taken" in a figure is as likely to be in one part as another *of equal area.* If, therefore, a moving line through A cuts CB in P and CD in R, the chance that XY produced cuts AB is $\int_0^1\frac{\text{CPR}}{\text{CBD}}\,d\,\frac{\text{CAR}}{\text{CAD}}$, the d depending on the motion of AR. Now

$$\frac{\text{CPR}}{\text{CBD}}=\frac{\text{CP}}{\text{CB}}\cdot\frac{\text{CR}}{\text{CD}},$$

or (completing the parallelogram ACBC′)

$$\frac{\text{CPR}}{\text{CBD}} = \frac{\text{CR}}{\text{C'R}} \cdot \frac{\text{CR}}{\text{CD}}, \text{ or } \left(\text{if } z = \frac{\text{CR}}{\text{CD}} = \frac{\text{CAR}}{\text{CAD}}\right) = \frac{z^2}{2-z}.$$

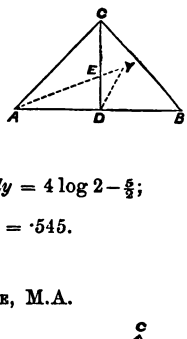
Therefore the chance that XY produced cuts AB is

$$\int_0^1 \frac{z^2}{2-z}\,dz, \text{ or (if } z = 2-u) = \int_1^2 \frac{(2-u)^2}{u}\,du = 4\log 2 - \tfrac{5}{2}.$$

As this depends on nothing that distinguishes A from B, the chance that XY produced cuts AB is its double, $8\log 2-5$, a little over ·545. Having tried two rough experiments, one by making dots at random, the other by taking the full stops in an equilateral triangle described on the top line of p. 304 of the *Educational Times*, I found that out of 208 and 80 XY's they gave respectively 91 and 50 intersections with the bases; therefore the values ·433 and ·625, of which the mean is ·529.

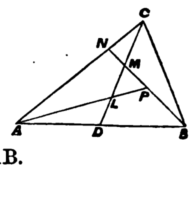
Although the value is independent of the dimensions of the figure when finite, the chance is unity when the sides are infinite and CD finite; but this is not more paradoxical than that under the same circumstances the area of CPR should vary as CR simply. If CA and CB are equal logarithmic curves, AB a common asymptote to them, and CD perpendicular to AB, the chance is independent of the scale of the curves, being always $\frac{1}{2}$ when this is finite. [For a more curious case, see Quest. 6170.]

II. *Solution by* T. R. Terry, M.A.; Prof. Johnson, M.A.; *and others.*



It is obviously sufficient to consider the case where CD is at right angles to DB and equal to DB. If Y be a fixed point in CDB, and AB $= 2a$, the chance that XY meets AB $= \dfrac{\Delta\text{EAD}}{\Delta\text{CAD}} = \dfrac{y}{a+x}$, where (x, y) are the coordinates of Y referred to DB and DC as axes; hence, for all positions of Y, the chance that XY meets AD is $\dfrac{2}{a^2}\displaystyle\int_0^a\int_0^{a-x} \frac{y}{a+x}\,dx\,dy = 4\log 2 - \tfrac{5}{2}$; therefore the chance that XY meets AB is $8\log 2 - 5 = \cdot 545$.

III. *Solution by* Professor Wolstenholme, M.A.



Let (X, Y) be the coordinates of a point P in the triangle BCD,

$$\left(\text{X} = \frac{\Delta\text{PDC}}{\Delta\text{BDC}},\ \ \text{Y} = \frac{\Delta\text{PDB}}{\Delta\text{CDB}}\right),$$

and let AP meet DC in L, and BP meet DC in M, and AC in N; then, if Q be a point in either of the triangles ALD or CMN, PQ will intersect the base AB.

The equation of AP is $\qquad \dfrac{y}{\text{Y}} = \dfrac{1+x}{1+\text{X}}$;

hence, at L, $y = \dfrac{\text{Y}}{1+\text{X}}$; hence the area of the triangle ADL $= \dfrac{\text{Y}}{1+\text{X}}$ (that of each of the triangles ADC, CDB being unity).

Again, the equation of BP is $\dfrac{y}{Y} = \dfrac{1-x}{1-X}$,

hence, at M, we have $y = \dfrac{Y}{1-X}$, therefore $\dfrac{CM}{CD} = \dfrac{1-X-Y}{1-X}$;

and, at N, $y = x+1$, therefore $\dfrac{y}{Y} = \dfrac{2-y}{1-X} = \dfrac{2}{1-X+Y}$,

therefore $$\frac{CN}{CA} = 1 - \frac{2Y}{1-X+Y} = \frac{1-X-Y}{1-X+Y};$$

also the triangle $CMN = \dfrac{(1-X-Y)^2}{(1-X)(1-X+Y)}$;

hence the chance of PQ meeting the base AB will be

$$\iint \left(\frac{Y}{1+X} + \frac{(1-X-Y)^2}{(1-X)(1-X+Y)}\right) dX\, dY \div \iint dX\, dY,$$

the limits of Y being 0 and $1-\alpha$, and of X, 0, 1. Now we have

$$\iint dX\, dY = \tfrac{1}{2}, \quad \iint \frac{Y}{1+X}\, dX\, dY = \tfrac{1}{2}\int \frac{(1-X)^2\, dX}{1+X} = \tfrac{1}{2}\int \frac{X^2}{2-X}\, dX$$

$$= \tfrac{1}{2}\int\left(\frac{4}{2-X} - 2 - X\right) dX = \tfrac{1}{2}\left(4\log 2 - \tfrac{5}{2}\right);$$

$$\iint \frac{(1-X-Y)}{(1-X)(1-X+Y)}\, dX\, dY = \int_0^1\int_0^x \frac{(X-Y)^2}{X(X+Y)}\, dX\, dY$$

$$= \iint \left(\frac{4X}{X+Y} - 4 + \frac{X+Y}{X}\right) dX\, dY = \int \left(4X\log 2 - 4X + \frac{3X}{2}\right) dX$$

$$= 2\log 2 - 2 + \tfrac{3}{4} = 2\log 2 - \tfrac{5}{4}.$$

Hence the chance required is $8\log 2 - 5 = \cdot 54517744$.

[The work might be simplified by noticing at first that the chance of PQ meeting AB must be just double that of meeting AD, and that this last chance is $$2\iint \frac{Y}{1+X}\, dX\, dY\,\Big].$$

6050. (By J. J. Walker, M.A.)—Prove that the circle with (ξ, η, ζ) as centre, and cutting the circle $S = 0$ orthogonally, is

$$(\xi \sin A + \ldots)\, S - (x \sin A + \ldots)\left(x\frac{dS}{d\xi} + \ldots\right) = 0,$$

ABC being the triangle of reference, and $\dfrac{dS}{d\xi}\ldots$ standing for the result of substituting ξ, η, ζ for x, y, z in $\dfrac{dS}{dx}\ldots$

Solution by E. W. Symons, B.A.; Christine Ladd; *and others.*

Since the new circle cuts S orthogonally, the radical axis of the circles is the polar of the centre (ξ, η, ζ) of the new circle with respect to S. Hence the equation of the new circle is of the form

$$S = \lambda (x \sin A + \dots)\left(x \frac{dS}{d\xi} \dots\right).$$

Expressing that (ξ, η, ζ) is the pole of the line $x \sin A + \dots = 0$ with respect to this circle, we have at once $\lambda = (\xi \sin A + \dots)^{-1}$; therefore, &c.

6007. (By J. Griffiths, M.A.)—Show that the conics $x^2+y^2+z^2-(l_1x+m_1y+n_1z)^2 = 0$, $x^2+y^2+z^2-(l_2x+m_2y+n_2z)^2=0$ (where $z=0$ is the line infinity, and x, y are rectangular axes) will intersect at an angle θ if

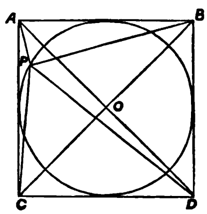
$$\frac{l_1l_2+m_1m_2+n_1n_2-1}{(1-l_1^2-m_1^2-n_1^2)^{\frac12}(1-l_2^2-m_2^2-n_2^2)^{\frac12}} = \cos\theta.$$

Solution by the Rev. F. D. Thomson, M.A.

The two curves in the question are the sections of the two right circular cones represented by the same equations in rectangular coordinates by a plane parallel to the plane of x, y. Now it is easily shown, by taking the traces of these cones on a concentric sphere, that the expression in the question gives the cosine of the angle between the tangent planes through a common generator of the cones, and therefore will not give the cosine of the angle between the traces of these planes upon a plane parallel to x, y, that is to say, between the tangents to the two plane curves at a common point.

5989. (By Professor Cochez.)—Sur la circonférence inscrite dans un carré on prend un point quelconque M duquel on voit les diagonales du carré sous des angles α et β; démontrer la relation $\tan^2\alpha+\tan^2\beta = 8$.

Solution by E. Anthony, M.A.; G. Turriff, M.A.; *and others.*



As vectors $PB = OB-OP$, $PC = -OB-OP$, therefore $PB \cdot PC = OP^2-OB^2+2V(OB \cdot OP)$.

Taking the vector and scalar parts, we get as lines $PC \cdot PB \sin\alpha = 2OP \cdot OB \sin POB$,

$$-PC \cdot PB \cos\alpha = OB^2-OP^2 = OP^2,$$

therefore $\tan^2\alpha = 8\sin^2 POB$;

similarly, $\tan^2\beta = 8\sin^2 POA = 8\cos^2 POB$,

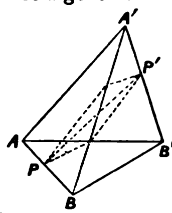
therefore $\tan^2\alpha+\tan^2\beta = 8$.

[Other proofs may be seen on pp. 59—61 of the *Diary* for 1852.]

6035. (By Professor TOWNSEND, F.R.S.)—A hyperbolic paraboloid being supposed to pass through the four sides of a skew quadrilateral in a space; show, on elementary principles, that the intercepted sheet of its surface bisects the volume of the tetrahedron determined by the four vertices of the quadrilateral.

I. *Solution by* Professor WOLSTENHOLME, M.A.

Let AA′B′B be the skew quadrilateral; then, if PP′ be a generator of the paraboloid of the same system as AA′ and BB′, we have AA′, PP′, BB′ all parallel to the same plane; and a plane through PP′ parallel to this fixed plane will cut the surface of the tetrahedron in a parallelogram of which PP′ is a diagonal, and if we then take a consecutive position of this plane, the elementary volume of the tetrahedron included will obviously be divided into two equal parts by the sheet of the paraboloid.



[Though Prof. WOLSTENHOLME states that he is not quite satisfied with this proof, it looks right, and may pass muster.

In general, the surface whose equation is $yz = kxw$, where k is positive, will divide the tetrahedron of reference into two parts which are in the ratio $k-1-\log k : \log k - 1 + k^{-1}$.]

II. *Solution by the* PROPOSER.

Denoting by P, Q, R, S the four vertices of the quadrilateral taken in consecutive order, and by p, q, r, s the four perpendiculars let fall from them upon the tangent plane at any element $d\alpha$ of the intercepted sheet of the surface; then, since, for every position of $d\alpha$ within the limits of the sheet, two opposite vertices P and R of the quadrilateral lie always at one side, and the remaining two Q and S at the other side of the tangent plane to the sheet, therefore, for the entire extent of its area, the two sums $\Sigma(pd\alpha)$ and $\Sigma(rd\alpha)$ are each equal to three times the entire tetrahedral volume at the PR side, and the two $\Sigma(qd\alpha)$ and $\Sigma(sd\alpha)$ each to three times the entire at the QS side of the area; and, in consequence, to prove the property, it remains only to show that, for the entire extent of the area, the four sums $\Sigma(pd\alpha)$, $\Sigma(qd\alpha)$, $\Sigma(rd\alpha)$, $\Sigma(sd\alpha)$ are all equal; which is easily done, as follows:—

Conceiving the entire area of the sheet divided into elementary parallelograms by two systems of equidistant parallel planes, one system parallel to the two opposite sides PQ and RS, and the other to the two QR and SP of the quadrilateral, and each consequently intersecting the sheet in the corresponding system of generators of the surface; then, since, for every pair of elements $d\alpha$ and $d\beta$, determined by a common pair of consecutive planes of the former system, and by two opposite pairs of consecutive planes equidistant from the extreme lines of the latter system, the two elementary volumes $pd\alpha$ and $qd\beta$, and also the two $rd\alpha$ and $sd\beta$, are manifestly equal; therefore, &c.

Otherwise thus, by the Integral Calculus:—Denoting by M the middle point of the diagonal PR of the quadrilateral, and supposing the tetrahedron divided into thin triangular slices by a system of planes parallel to that of the bisecting triangle QMS; then, the latter being manifestly a system of diametral planes of the paraboloid, intersects consequently the surface

in a system of equal parabolas, all having the same parameter as that of its section by the determining plane QMS, and each dividing the corresponding slice of the solid into two segments, the sum of which at one side of the sheet is to be shown equal to the sum at the other side for the entire solid; which is easily done as follows:—

Denoting, for either half of the solid as bisected by the plane QMS, by x the distance of any plane of section from the corresponding vertex, P suppose, of the half, and by h that of the plane QMS itself from the same; we have manifestly, for the tetrahedral and paraboloidal volumes U and V corresponding to that half of the solid, the values

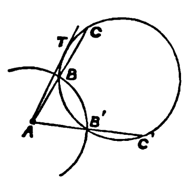
$$U = \int_0^h \mu x^2 dx = \tfrac{1}{3}\mu h^3, \text{ and } V = \int_0^h \nu x^3 dx = \tfrac{1}{4}\nu h^4,$$

where μ and ν are constants depending on the magnitude and form of the half; but, the two sides QM and SM of the terminal triangle QMS of the half being manifestly tangents to the parabolic section of the surface by its plane, and the parabolic being consequently two-thirds of the triangular area in that plane, therefore $\nu h = \frac{2}{3}\mu$; hence $V = \frac{1}{6}\mu h^3 = \frac{1}{2}U$, and, since similarly for the remaining half of the solid, $V' = \frac{1}{2}U'$; therefore, &c.

5658. (By Byomakesa Chakravarti.)—From a given point A, draw a straight line ABC, cutting a given circle in B and C, so that the whole line shall be bisected in B; and show what are the conditions in order that the problem may be possible.

Solution by H. L. Orchard, B.A., L.C.P.; J. O'Regan; *and others.*

From the given point A, draw a straight line AT touching the given circle in T.



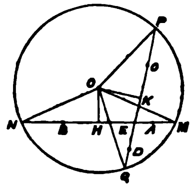
With centre A and radius $AB = \frac{1}{2}\sqrt{2} \cdot AT$, draw a circle cutting the given circle in B and B′, and let the straight lines AB, AB′ produced, cut that circle again in C, C′ respectively. Either of the straight lines ABC, AB′C′ is the line required. For $AC \cdot AB = AT^2 = 2AB^2$, therefore $AC = 2AB$; *i.e.*, ABC is bisected in B.

If the circles touch, the two lines ABC, AB′C′ coincide. If the circles do not meet, the solution is impossible. The distance of the given point from the given circle must not be greater than the diameter of that circle.

5931. (By Professor Matz, M.A.)—If A, B, C, D are four points taken at random in the surface of a given circle; show that the chance that E, the intersection of the straight lines through A, B and C, D, lies between A, B and between C, D, is $\dfrac{1}{3} - \dfrac{35}{36\pi^2}$.

Solution by Professor SEITZ, M.A.

Let MN and PQ be the chords through A, B and C, D, and OH and OK the perpendiculars from O, the centre of the circle, upon MN and PQ.



Let $OM = r$, $EA = w$, $AB = x$, $EC = y$, $CD = z$, $EM = u_1$, $EN = u_2$, $EP = v_1$, $EQ = v_2$, $\angle MOH = \theta$, $\angle POK = \phi$, $\angle MEP = \psi$, and μ = the angle MN makes with some fixed line.

Then we have $OH = r\cos\theta$, $OK = r\cos\phi$, $MH = r\sin\theta$, $PK = r\sin\phi$, $u_1 + u_2 = 2r\sin\theta$, $v_1 + v_2 = 2r\sin\phi$, and

$$u_1 u_2 = v_1 v_2 = r^2 \operatorname{cosec}^2\psi\,(\sin^2\psi + 2\cos\theta\cos\phi\cos\psi - \cos^2\theta - \cos^2\phi).$$

An element of the circle at A is $r\sin\theta\,d\theta\,dw$, at B it is $x\,dx\,du$, at C it is $r\sin\phi\,d\phi\,dy$, and at D it is $z\,dz\,d\psi$.

That E may lie between A, B and between C, D, the limits of θ must be 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ, and doubled; of ψ, $\theta - \phi$ and $\theta + \phi$, and doubled; of μ, 0 and 2π; of w, 0 and u_1, and doubled; of x, w and $w + u_2$; of y, 0 and v_1, and doubled; and of z, y and $y + v_2$.

Hence, since the whole number of ways the four points can be taken is $\pi^4 r^8$, the required chance is

$$\frac{16}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\int_0^{2\pi}\int_0^{u_1}\int_w^{w+u_2}\int_0^{v_1}\int_y^{y+} r\sin\theta\,d\theta\,r\sin\phi\,d\phi\,d\psi\,d\mu\,dw\,x\,dx\,dy\,z\,dz$$

$$= \frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\int_0^{2\pi}\int_0^{u_1}\int_w^{w+u_2}\int_0^{v_1}\left\{(y+v_2)^2 - y^2\right\} \times \sin\theta\sin\phi\,d\theta\,d\phi\,d\psi\,d\mu\,dw\,x\,dx\,dy$$

$$= \frac{16}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\int_0^{2\pi}\int_0^{u_1}\int_w^{w+u_2} v_1 v_2 \sin\theta\sin^2\phi\,d\theta\,d\phi\,d\psi\,d\mu\,dw\,x\,dx$$

$$= \frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\int_0^{2\pi}\int_0^{u_1}\left\{(w+u_2)^2 - w^2\right\} v_1 v_2 \sin\theta\sin^2\phi\,d\theta\,d\phi\,d\psi\,d\mu\,dw$$

$$= \frac{16}{\pi^4 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\int_0^{2\pi} u_1 u_2 v_1 v_2 \sin^2\theta\sin^2\phi\,d\theta\,d\phi\,d\psi\,d\mu$$

$$= \frac{32}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}(\sin^2\psi + 2\cos\theta\cos\phi\cos\psi - \cos^2\theta - \cos^2\phi)^2 \times \sin^2\theta\sin^2\phi\operatorname{cosec}^4\psi\,d\theta\,d\phi\,d\psi$$

$$= \frac{64}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}(3\phi - 3\sin\phi\cos\phi - 2\sin^3\phi\cos\phi)\sin^2\theta\sin^2\phi\,d\theta\,d\phi$$

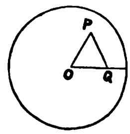
$$= \frac{16}{9\pi^3}\int_0^{\frac{1}{2}\pi}(9\theta^2 - 18\theta\sin\theta\cos\theta + 9\sin^2\theta - 9\sin^4\theta - 4\sin^6\theta)\sin^2\theta\,d\theta$$

$$= \frac{1}{3} - \frac{35}{36\pi^2}.$$

6056. (By Professor SEITZ, M.A.)—A triangle is formed by joining two random points within a circle to the centre and with each other; show that the chance that the circle circumscribing the triangle lies wholly within the given circle is $\frac{3}{8}$.

I. *Solution by* T. R. TERRY, M.A.; DR. HART; *and others.*

Let P, Q be the points inside the circle of radius unity, OP $= x = \sin\phi$, OQ $= y$, $\angle$POQ $= \theta$. The total number of positions of P and Q is π^2; and clearly Q may be supposed to be always on the initial line. If we take number of positions of Q between distances y and $y+dy$ from O to be $2\pi y\,dy$, we have, for the required probability,



$$p = \frac{1}{\pi^2}\iint x\,dx\,d\theta \int 2\pi y\,dy,$$

the limits being such that $(x^2+y^2-2xy\cos\theta)^{\frac{1}{2}} < \sin\theta$;

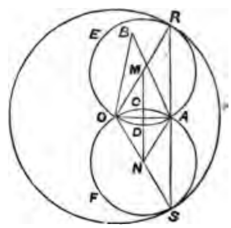
that is, y does not lie between $\sin(\phi-\theta)$ and $(\phi+\theta)$ $(y_1+y_2$ say). Thus we must integrate from $y=0$ to $y=y_1$, and then from $\theta=0$ to $\theta=\phi$; and also from $y=y_2$ to $y=1$, and then from $\theta=0$ to $\theta=\pi-\phi$, and then double the sum, for P may be below OQ. Hence

$$p = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin\phi\cos\phi\,d\phi\left\{\int_0^{\phi}\sin^2(\phi-\theta)\,d\theta+\int_0^{\pi-\phi}\sin^2(\theta+\phi)\,d\theta\right\}$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\sin\phi\cos\phi\,d\phi\,(\pi+\sin 2\phi) = \tfrac{3}{8}.$$

II. *Solution by the* PROPOSER.

Let A and B be the two random points, and O the centre of the given circle. Through A draw RS perpendicular to OA; on the radii OR and OS draw circles, with centres M and N. Join MN, cutting the circles in C and D. Now it is evident that, if B be taken anywhere in the surface AREOCA, or ASFODA, the circle circumscribing the triangle AOB will lie wholly within the given circle.



Let OR $= r$, OA $= x$, and $\angle$ORA $= \theta$;

then $x = r\sin\theta$, $dx = r\cos\theta\,d\theta$, area ACODA $= \frac{1}{2}r^2(\theta-\sin\theta\cos\theta)$,

area AREOCA + area ASFODA = area circle M + area circle N
-2 area ACODA $= \frac{1}{2}\pi r^2 - r^2(\theta-\sin\theta\cos\theta)$.

The limits of x are 0 and r, and those of θ are 0 and $\frac{1}{2}\pi$. Hence, since the whole number of ways the two points can be taken is $\pi^2 r^4$, we have

$$p = \frac{1}{\pi^2 r^4}\int_0^r\left\{\tfrac{1}{2}\pi r^2 - r^2(\theta-\sin\theta\cos\theta)\right\}2\pi x\,dx$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}(\pi-2\theta+2\sin\theta\cos\theta)\sin\theta\cos\theta\,d\theta = \tfrac{3}{8}.$$

6113. (By D. EDWARDES.)—Prove that

$$50\left\{(x-y)^7+(y-z)^7+(z-x)^7\right\}^2 =$$
$$49\left\{(x-y)^4+(y-z)^4+(z-x)^4\right\}\left\{(x-y)^5+(y-z)^5+(z-x)^5\right\}^2.$$

Solution by E. W. SYMONS, B.A.; T. R. TERRY, M.A.; *and others.*

Let $y-z$, $z-x$, $x-y$ be roots of $X^3+p_2X+p_3=0$, then, applying NEWTON's theorem on sums of powers of roots, we have $s_1=0$, $s_2=-2p_2$, $s_3=-3p_3$, $s_4=2p_2^2$, $s_5=5p_2p_3$, $s_6=3p_3^2-2p_2^3$, $s_7=7p_2^2p_3$; therefore

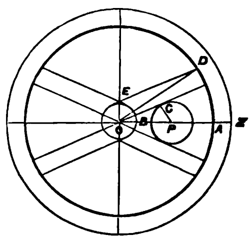
$$50s_7^2 = 50\,.\,49p_2^4\,.\,p_3^2 = 49\,.\,2p_2^2\,.\,25p_2^2\,.\,p_3^2 = 49s_4\,.\,s_5^2,$$

therefore, &c.

6073. (By the EDITOR.)—Into an empty tub two coins are thrown at random; show that the probability that both the coins will rest on the same diameter of the circular bottom of the tub is $\dfrac{1}{\pi}\left\{2(\alpha+\beta)+\sin 2(\alpha+\beta)\right\}$, where 2α, 2β are the angles that the coins subtend at the centre of the bottom of the tub when placed close to the side, and the coins are supposed to be of such a size as to make $\alpha+\beta<\frac{1}{2}\pi$, that is to say, small enough to allow of the possibility of failure.

I. *Solution by* Professor NASH, M.A.; G. HEPPEL, M.A.; *and others.*

Let r be the radius OZ of the bottom of the tub; a and b the radii PC and OB ($=$AZ) respectively of the coins; x the distance OP of the centre of the coin (a) from the centre O of the bottom; AD a circle with O as centre and OA as radius; 2θ the angle which the coin (a) subtends at O; then



$$\sin\alpha = \frac{a}{r-a},\quad \sin\beta = \frac{b}{r-b},\quad \sin\theta = \frac{a}{x}.$$

In the case represented, the coins will meet the same diameter if the distance of the centre of the coin (b) from the tangents from O to the coin (a) be less than b, that is to say, if the centre of the coin (b) lie within the area OADE, or within the similar figures in the other quadrants. If, however, the angle AOD ($=\theta+\beta$) be greater than $\frac{1}{2}\pi$, the coins must meet the same diameter, whatever be the position of the coin (b), and this is the case if $x<a\sec\beta$. Now we have

$$\text{Area OADE} = \tfrac{1}{2}(\theta+\beta)(r-b)^2+\tfrac{1}{2}b(r-b)\cos\beta-\tfrac{1}{2}b^2\tan\theta,$$

therefore, for this position of P, the probability is

$$\frac{1}{\pi}\left\{2(\theta+\beta)+\sin 2\beta-2\sin^2\beta\tan\theta\right\},$$

and the probability of this value of x is $\dfrac{2\pi x\,dx}{\pi(r-a)^2} = -\dfrac{2\sin^2\alpha\cos\theta\,d\theta}{\sin^3\theta}$;

hence the whole probability required is

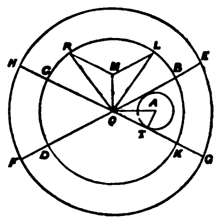
$$\frac{\pi a^2}{\pi (r-a)^2 \cos^2\beta} - \frac{2\sin^2\alpha}{\pi}\int_{\frac{1}{2}\pi-\beta}^{\alpha} (2\beta + \sin 2\beta - 2\sin^2\beta \tan\theta + 2\theta) \frac{\cos\theta\, d\theta}{\sin^3\theta}$$

$$= \frac{\sin^2\alpha}{\cos^2\beta} + \frac{\sin^2\alpha}{\pi}\left[(2\beta + \sin 2\beta)\frac{1}{\sin^2\theta} - 4\sin^2\beta\cot\theta + \frac{2\theta}{\sin^2\theta} + 2\cot\theta\right]_{\frac{1}{2}\pi-\beta}^{\alpha}$$

$$= \frac{1}{\pi}\left\{\frac{\pi\sin^2\alpha}{\cos^2\beta} + (2\beta + \sin 2\beta)\left(1 - \frac{\sin^2\alpha}{\cos^2\beta}\right) + \sin 2\alpha\cos 2\beta \right.$$
$$\left. - 2\tan\beta\cos 2\beta\sin^2\alpha + 2\alpha - \frac{(\pi - 2\beta)\sin^2\alpha}{\cos^2\beta}\right\}$$

$$= \frac{1}{\pi}\left\{2(\alpha+\beta) + \sin 2(\alpha+\beta)\right\}.$$

II. *Solution by* Prof. Seitz, M.A.; A. Martin, M.A.; *and others.*



Let EHFG represent the bottom of the tub; O its centre; A the centre of the coin which subtends at O the angle 2α; EF, GH diameters touching the circle A; EB equal to the radius of the coin which subtends at O the angle 2β; BCDK a circle drawn around O as centre with OB as radius; and ML, MR parallels to OB, OC respectively, and at a distance equal to EB from these lines.

Now, if the centre of the second coin fall anywhere on the semicircle KBC, except on the surface LMR, the coins will rest on the same diameter. The number of favourable positions for the semicircle CDK is the same as that for KBC.

Let $OE = r$, $AI = b$, $EB = c$, $AO = x$, $\angle AOT = \theta$, $\angle LOB = \beta$,

and area circle BCDK $-$ 2 area LMR $= u$.

Then we have $OB = r - c$, $OM = c\sec\theta$, $x = b\operatorname{cosec}\theta$,

$dx = -b\cos\theta\operatorname{cosec}^2\theta\, d\theta$, $\angle LOM = \frac{1}{2}\pi - \theta - \beta$, $\sin\alpha = \dfrac{b}{r-b}$, $\sin\beta = \dfrac{c}{r-c}$,

circle KBCD $= \pi(r-c)^2$, $u = 2(r-c)^2\{\theta + \beta + \sin\beta\sec\theta\cos(\theta+\beta)\}$.

From $x = 0$ to $x = b\sec\beta$ the number of favourable positions of the second coin for each value of x is $\pi(r-c)^2$, and from $x = b\sec\beta$ to $x = r - b$ the number of favourable positions is u; hence the required chance is

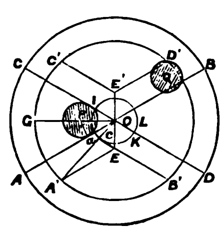
$$p = \left\{\int_0^{b\sec\beta} \pi(r-c)^2 x\,dx + \int_{b\sec\beta}^{r-b} ux\,dx\right\} \div \int_0^{r-b} \pi(r-c)^2 x\,dx$$

$$= \frac{\sin^2\alpha}{\cos^2\beta} + \frac{4\sin^2\alpha}{\pi}\int_{\alpha}^{\frac{1}{2}\pi-\beta} \{\theta + \beta + \sin\beta\sec\theta\cos(\theta+\beta)\}\cos\theta\operatorname{cosec}^3\theta\, d\theta$$

$$= \frac{1}{\pi}\{2(\alpha+\beta) + \sin 2(\alpha+\beta)\}.$$

III. *Solution by* S. Watson; Prof. Matz, M.A.; *and others.*

Let ACD be the tub's bottom, of centre O and radius r; P, Q the coins, of radii r_1, r_2. Draw, with radius $r-r_2$, the circle A'B'C' concentric with the bottom; through O draw the tangents AB, CD to the coin P; and parallel to those tangents, at the distance r_2 from them, draw A'E, B E, C'E', D'E'. Join OP, producing it to meet the circle A'B'C' in G; also join OA', EE' passing through O; and draw Pa perpendicular to OA, and Ec to OA'. Put



$\mathrm{OP} = x$, $\angle \mathrm{POA} = \theta$, and $\angle \mathrm{AOA'} = a$.

Then $\angle \mathrm{GOE} = 90°$, therefore $r_2 = \mathrm{OE}\cos\theta$,

and $\mathrm{E}c = \mathrm{OE}\cos(a+\theta) = r_2(\cos a - \sin a \tan\theta)$,

therefore $\triangle \mathrm{A'OE} = \frac{1}{2}\mathrm{A'O}\,.\,\mathrm{E}c = \frac{1}{2}r_2(r-r_2)\left(\cos a - \dfrac{r_1 \sin a}{\sqrt{x^2-r_1^2}}\right).$

Also, $$\text{sector A OG} = \tfrac{1}{2}(r-r_2)^2\left(a + \sin^{-1}\frac{r_1}{x}\right),$$

therefore $$\text{area A'EB'D'E'C'} = 4\,(\text{sector A'OG} + \triangle \mathrm{A'OE})$$
$$= 2(r-r_2)\left(m + (r-r_2)\sin^{-1}\frac{r_1}{x} - \frac{r_1 r_2 \sin a}{\sqrt{x^2-r_1^2}}\right) \text{......(1)},$$

where $$m = (r-r_2)\,a + r_2\cos a.$$

Now, when $\theta = 90° - a$, or $\mathrm{OP} = r_1 \sec a$, the points A', B', as also C', D', will coincide; hence, if with radius $\mathrm{OI} = r_1\sec a$, we draw the circle ILK, then, so long as the centre of P falls within this circle, the coins will lie upon a diameter, wherever Q may lie; and the chance of this happening is
$$\frac{\text{circle ILK}}{\pi(r-r_1)^2} = \frac{r_1^2\sec^2 a}{(r-r_1)^2} \text{ .... (2).}$$

Again, that the centre of P may fall on a ring of breadth dx, the chance is $\dfrac{2\pi x\,dx}{\pi(r-r_1)^2}$; and that the centre of Q may lie in the area A'B'D'C', which it must do in order that the coins may lie upon a diameter, the chance is $\dfrac{\text{area A'B'D'C'}}{\pi(r-r_2)^2}$; hence, taking the product of those chances and integrating, we have, by (1),
$$\frac{2}{(r-r_1)^2(r-r_2)^2\pi}\int_{r_1\sec a}^{r-r_1} \text{area A'B'D'C'}\,.\,x\,dx$$
$$= \frac{2}{\pi}\left\{\frac{m}{r-r_2} + \sin^{-1}\frac{r_1}{r-r_1} - \frac{\sin^{-1}(\cos a)\,r_1^2}{(r-r_1)^2\cos^2 a}\right.$$
$$\left. + \frac{r_1\cos^2 a\,(r-r_2-2r\cos a)\,(\sqrt{r^2-2rr_1}-r_1\tan a)-r_1^2}{(r-r_1)^2(r-r_2)^2\cos^2 a}\right\} \text{ ...(3).}$$

This is the chance when the centre of P falls without the circle ILK; hence the sum of (2) and (3) is the required chance. [This leads to the result stated in the Question. The Proposer's solution, with an illustrative numerical example, is given on p. 73 of Vol. IV. of the *Reprint*.]

6095. (By Professor TOWNSEND, F.R.S.)—The planes of two conics situated on a common cone of revolution being supposed to intersect in the equatoreal plane of the cone; show that each conic is the image of the other by central refraction through a thin spherical lens, of appropriate power, having its optical centre at the vertex of the cone.

Quaternion Solution by J. J. WALKER, M.A.

Taking the vertex of the cone (O) as origin, let γ be a (unit) vector in the direction of the axis, while α, β are the vectors of any two points in the line of intersection of the planes of the conics, which planes cut the axis at the points (C) $m\gamma$, (C′)$n\gamma$. Then ρ, σ being sides of the cone to points P, S on the two conics respectively, and θ the angle of the cone,

$$\rho = (x\alpha + y\beta + zm\gamma) \div (x + y + z) \quad \text{.....................(1)},$$

$$\sigma = (x'\alpha + y'\beta + z'n\gamma) \div (x' + y' + z') \quad \text{.................(2)},$$

whence $S\rho\gamma = -zm \div (x+y+z)$, $S\sigma\gamma = -z'n \div (x'+y'+z')$;

but $S\rho\gamma = -2\cos\theta\, T\rho$, $S\sigma\gamma = -2\cos\theta\, T\sigma$,

so that,

$$1 \div T\rho - 1 \div T\sigma = 2\cos\theta \left\{(x+y+z) \div zm - (x'+y'+z') \div z'n\right\}.$$

If O, P, S are collinear, $x : zm = x' : z'n$, $y : zm = y' : z'n$,

and

$$\frac{1}{\text{OP}} - \frac{1}{\text{OS}} = 2\cos\theta \cdot \left(\frac{1}{\text{OC}} - \frac{1}{\text{OC}'}\right).$$

Thus it appears that the same lens would serve for any two sections intersecting in the equatoreal plane, and cutting in given points the axis of the cone. Assuming, as has been done above, that C, C′, and consequently P, S, are on the same side of the vertex, a thin convexo-concave lens of refractive index $\mu = 1 + 2\cos\theta$, and having the radii of its surfaces equal to OC, OC′, would be appropriate.

544?. (By J. C. MALET, M.A.)—Given two conics U and V, show (1) that we can determine four points such that any fourth conic (W) through them shall possess the property, that, if from any point on W we draw one tangent to U and another to V, then the line joining the points of contact envelops another conic; and (2) deduce therefrom, as a particular case, the theorem in Question 5395; also prove (3) as a particular case, that if from any point on a common chord of two conics we draw one tangent to each of the conics, the line joining the points of contact passes through one of two fixed points.

Solution by the PROPOSER.

The four points in question are the four intersections of the conics U and V and the problem may be stated as follows:—

"If from any point on a conic W, which passes through the intersections of the two conics U and V we draw one tangent to U and another to V, then the line joining the points of contact envelops a fourth conic."

Let the equations of U and V be respectively

$$ax^2+by^2-z^2=0, \quad a_1x^2+b_1y^2-z^2=0;$$

then the condition that the line joining the point

$$\left(\frac{\cos\alpha}{a^{\frac{1}{2}}}, \frac{\sin\alpha}{b^{\frac{1}{2}}}, 1,\right) \text{ on U, to the point } \left(\frac{\cos\beta}{a_1^{\frac{1}{2}}}, \frac{\sin\beta}{b_1^{\frac{1}{2}}}, 1\right) \text{ on V,}$$

should touch the conic $\dfrac{x^2}{A_1}+\dfrac{y^2}{B_1}+\dfrac{z^2}{C_1}=0,$

is $$A_1\left\{\frac{\sin\alpha}{b^{\frac{1}{2}}}-\frac{\sin\beta}{b_1^{\frac{1}{2}}}\right\}+B_1\left\{\frac{\cos\alpha}{a^{\frac{1}{2}}}-\frac{\cos\beta}{a_1^{\frac{1}{2}}}\right\}^2$$
$$+C_1\left\{\frac{\cos\alpha\sin\beta}{(ab_1)^{\frac{1}{2}}}-\frac{\cos\beta\sin\alpha}{(a_1b)^{\frac{1}{2}}}\right\}^2=0,$$

while the condition that the tangents to U and V at these points should meet on the conic $Ax^2+By^2+Cz^2=0$

is $$A\,(b^{\frac{1}{2}}\sin\alpha-b_1^{\frac{1}{2}}\sin\beta)^2+B\,(a^{\frac{1}{2}}\cos\alpha-a_1^{\frac{1}{2}}\cos\beta)^2$$
$$+C\left\{(ab_1)^{\frac{1}{2}}\cos\alpha\sin\beta-(a_1b)^{\frac{1}{2}}\cos\beta\sin\alpha\right\}^2=0.$$

These conditions are identical, proving the following conditions are true,

$$A_1=Abb_1, \quad B_1=Baa_1, \quad C_1=Caa_1bb_1, \quad C=\frac{A(b-b_1)-B(a-a_1)}{ab_1-a_1b}.$$

Now the equation $(ab_1-a_1b)(Ax^2+By^2)+\{A\,(b-b_1)-B\,(a-a_1)\}z^2=0$ may be written $(Ab_1-Ba_1)(ax^2+by^2-z^2)-(Ab-Ba)(a_1x^2+b_1y^2-z^2)=0,$ and is therefore the most general equation of a conic passing through the intersections of U and V; hence (1) is proved. To deduce as a particular case the theorem in Question 5395, we need only remark that, if two tangents, one to each of two confocal conics, cut orthogonally, the locus of their intersection is a circle, which, as confocals cut at right angles, must pass through the intersections real or imaginary of the confocal conics.

(3) follows at once from the general case, for, whenever the equation $Ax^2+By^2+Cz^2=0$ represents two lines, then the tangential equation $Abb_1\lambda^2+Baa_1\mu^2+Caa_1bb_1\nu^2=0$ represents two points.

If we reciprocate the general theorem, we get the following:—"If three conics U, V, and W be touched by the same four lines, then, if one of the intersections with U of any tangent to W be A, and one of the intersections with V of the same line be B, the locus of the intersections of tangents to U and V at the points A and B is another conic."

As a particular case, we may take U, V, and W as three confocal conics, or, more particularly still, we get the following:—"If any line through one of the common foci of two confocal conics cut the conics respectively in A and B, then the intersection of tangents to the conics at these points lies on one of two fixed right lines.

6083. (By E. W. Symons, B.A.)—If $lx+my+nz=p$ touch a surface, prove that the coordinates of the point of contact are $\dfrac{dp}{dl}, \dfrac{dp}{dm}, \dfrac{dp}{dn}$, pro-

where l, m, n are direction-cosines, and p is made homogeneous of the degree [illegible] in l, m, n.

Solution by [illegible], M.A.; [illegible], M.A.; *and others.*

What we want is the point of contact with the envelope of

$lx + my + nz - p = 0$, l, m, n being connected by $l^2 + m^2 + n^2 = 1$ .....(1, 2).

Following then the method of undetermined multipliers, we have that,

for some value of λ, $\lambda l = x - \frac{dp}{dl}$, $\lambda m = y - \frac{dp}{dm}$, $\lambda n = z - \frac{dp}{dn}$

simultaneously. Multiplying these by l, m, n respectively, and adding, we

find, using (1) and (2), that $\lambda = p - l\frac{dp}{dl} - m\frac{dp}{dm} - n\frac{dp}{dn} = 0$,

if, as is supposed, p be so prepared before differentiation, by means of (2), as to be homogeneous and of the first degree. This being so, it follows that

$$x - \frac{dp}{dl} = 0, \quad y - \frac{dp}{dm} = 0, \quad \text{and} \quad z - \frac{dp}{dn} = 0.$$

5587. (By E. W. Symons, B.A.)—Prove that the vertices of sections of the conicoid $ax^2 + by^2 + cz^2 = 1$, made by planes parallel to $lx + my + nz = 0$, all lie on the cone

$$\left\{ l(my + nz) - \left(\frac{m^2}{b} + \frac{n^2}{c}\right) ax \right\} (bny - cmz) + \ldots + \ldots = 0.$$

Solution by W. J. C. Sharp, M.A.; *the* Proposer; *and others.*

The vertices of a section are points at which the normals pass through the centre of the section. Now the equations of the normal to a section made by the plane $l\xi + m\eta + n\zeta = p$ are

$$(\xi - x)(bny - cmz) + \ldots + \ldots = 0, \quad (l\xi + \ldots + \ldots = p) \ldots\ldots\ldots (1);$$

and the coordinates of the centre of the section are easily found to be

$$\frac{pl}{a}\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)^{-1} \ldots\ldots;$$

therefore, writing $lx + my + nz$ for p, and expressing that (1) passes through the centre of the section, we get

$$\left\{ l(my + nz) - ax\left(\frac{m^2}{b} + \frac{n^2}{c}\right) \right\} (bny - cmz) + \ldots + \ldots = 0; \quad \text{therefore \&c.}$$

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CORRIGENDA.

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Page 60, line 26, *for* these *read* there.
Page 82, line 5 from bottom, *for* 2668 *read* 6268.

VOL. XXXII.

Page 72, *omit* from line 5 from bottom to line 8 from bottom of page 73; the paragraph to be left out belonging to the Solution of the closely related Question 5275, by the same author (Vol. XXXI., p. 27.)

*** Of this series thirty-three volumes have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

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Selby, A. L., Leigh, Manchester.
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Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham.
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Young, John, B.A.; Academy, Londonderry.

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Mathematical Papers, &c.

Solved Questions.

[illegible]

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6155. (By Professor CAYLEY, F.R.S.) — Given, by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance.

N.B.—If λ, μ, ν are the inclinations of a line to three rectangular axes, and α, β, γ the coordinates to the same axes of a point on the line, then the metrical coordinates of the line are

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h, \\ = \cos\lambda, & \cos\mu, & \cos\nu, & b\cos\nu - c\cos\mu, & c\cos\lambda - a\cos\nu, & a\cos\mu - b\cos\lambda, \end{array}$$

satisfying identically the relations $a^2+b^2+c^2 = 1$, $af+bg+ch = 0$.

Solution by Professor GENESE, M.A.

1. Let θ be the angle between the given straight lines; then

$$\cos\theta = aa' + bb' + cc', \text{ and } \sin^2\theta = L^2 + M^2 + N^2,$$

where $$L = \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}, \quad M = \begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}, \quad N = \begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}.$$

2. The equations to the straight lines are

$$(bz-cy = f, \quad cx-az = g, \quad ay-bx = h) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$(b'z-c'y = f', \quad c'x-a'z = g', \quad a'y-b'x = h') \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equations to a plane through (1), parallel to (2), will be found (using $af+bg+ch = 0$) to reduce to $Lx+My+Nz = -(a'f+b'g+c'h)$.........(3).

Therefore, the equation to plane through (2), parallel to (1), is

$$Lx+My+Nz = af'+bg'+ch' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The shortest distance between (1) and (2) is, therefore,

$$(af'+a'f+bg'+b'g+ch'+c'h) \div (L^2+M^2+N^2)^{\frac{1}{2}}.$$

3. Let A, B, C, F, G, H be the coordinates of this distance; then

$$A : B : C = L : M : N;$$

therefore $\quad A = L \operatorname{cosec} \theta, \quad B = M \operatorname{cosec} \theta, \quad C = N \operatorname{cosec} \theta.$

Again, the condition that ABCFGH should intersect (1), is found by equating to zero the expression for their shortest distance; hence

$$aF + bG + cH + fA + gB + hC = 0,$$

and
$$a'F + b'G + c'H + f'A + g'B + h'C = 0;$$

also
$$AF + BG + CH = 0;$$

and from these equations we can obtain F, G, H.

Let
$$P \equiv \begin{vmatrix} f, & g, & h \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}, \quad P' \equiv \begin{vmatrix} f', & g', & h' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix};$$

then
$$F = \frac{P(a' \cos \theta - a) + P'(a \cos \theta - a')}{\sin^2 \theta}, \text{ \&c.}$$

2782. (By Professor Crofton, F.R.S.)—(1) A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle. Also (2) three given points are known to be within a certain circle, which is otherwise altogether unknown: determine the most probable position of its centre. Again, (3) two given points are known to be within a circle, and a third given point is known to be outside it: determine the most probable position of its centre.

Solution by the Proposer.

The answer to (2) is, the centre of the minimum circle which satisfies the conditions, that is to say, the circumscribed circle of the triangle ABC, if the triangle is acute; if obtuse, then the circle on longest side as diameter: for, whatever be the radius of the circle, this point is a possible position for its centre; this is the case for no other point.

In (3) the centre is most likely to be at an infinite distance. If, however, the radius is not to exceed a certain limit, there must be a definite solution, though not easy to find.

[A Solution by Professor Wolstenholme is given on p. 28 of Vol. XII. of the *Reprint*.]

4143. (By the late Professor Clifford, F.R.S.)—Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings.

Solution by A. MARTIN, M.A.; Prof. MATZ, M.A.; *and others.*

Let T_1, T_2, T_3 be the tensions of the strings fastened at A, B, C respectively; λ_1, λ_2, λ_3 their respective moduli of elasticity; W the weight, and r the radius of the sphere; and R the radius of the circumscribing circle of the triangle; then, we have

$$R(1+\lambda_1 T_1) = (R^2-r^2)^{\frac{1}{2}} + r\sin^{-1}\frac{r}{R};$$

therefore
$$\lambda_1 = \frac{(R^2-r^2)^{\frac{1}{2}} - R + r\sin^{-1}\frac{r}{R}}{RT} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

For the equilibrium of the sphere, resolving forces vertically, we have

$$(T_1+T_2+T_3)\frac{r}{R} = W \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Resolving forces perpendicular to AB, in the plane of ABC, we have

$$(T_2+T_3)\cos A = T_1\cos(B-C) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Eliminating T_2+T_3 from (2), (3), $T_1 = \frac{WR}{r}, \ \frac{\cos A}{\cos A+\cos(B-C)} \ldots(4).$

Substituting in (1), we obtain

$$\lambda_1 = r\left(\frac{(R^2-r^2)^{\frac{1}{2}} - R + r\sin^{-1}\frac{r}{R}}{WR^2}\right)\left\{1+\frac{\cos(B-C)}{\cos A}\right\}, \quad \lambda_2 = \&c., \lambda_3 = \&c.$$

6153. (By Sir JAMES COCKLE, F.R.S.)—When $y+z=v$ and $x+v=w$; then $v=0$ and $w=0$ each solve $(x+y)(1+p+q)+zp = 0 \ldots\ldots\ldots\ldots(1)$. Show, without recourse to the general primitive of (1), the distinct nature, and, by means of that primitive, the distinct origin, of the two solutions; and that, taking $u=a$ and $v=b$ as independent particular integrals of (1), the Jacobian of u, v, w does not vanish identically.

Solution by the PROPOSER.

From $v=0$ we get $p=0$ and $1+q=0$, relations which reduce (1) to $0=0$, and show that $v=0$ is an integral capable of being completed into $v=b$. From $w=0$ we get $1+p=0$ and $1+q=0$, relations which reduce (1) to $-w=0$, not to $0=0$, and $w=0$ is a solution incapable of a similar completion. The integrating system of (1) is

$$dx : w = dy : (x+y) = -dz : (x+y) = dM : (-2M),$$

where M is JACOBI's multiplier. Hence $dw : w = -dM : 2M$ and $dv = 0$; whence $v=b$, and a value of M is w^{-2} or $(x+b)^{-2}$. Integrating,

$$(x+b)^{-2}\{(x+y)\,dx - (x+b)\,dy\} = 0;$$

and, after the integration, replacing $x+b$ by w, we have

$$\frac{z}{w} + \log w = a, \text{ or, say, } u = a.$$

The general primitive of (1) is $u = \phi v$; and $w = 0$ gives $u^{-1} = 0$, since $w \log w$ vanishes with w. Putting $\frac{du}{dx} = \lambda$, the Jacobian of u, v, w is

$$\begin{vmatrix} \lambda, & \lambda, & \lambda + \frac{1}{w} \\ 0, & 1, & 1 \\ 1, & 1, & 1 \end{vmatrix}, = -\frac{1}{w}, \text{ and } = \infty \text{ when } w = 0.$$

But, according to Boole (implicitly at pp. 331, 332 of his *Differential Equations*, 1865, and explicitly at pp. 60, 61 of the *Supplement*), this determinant ought to vanish identically.

5713. (By Professor Sylvester, F.R.S.)—It is a well-known theorem of Euler's that the number of ways in which x can be formed by putting together j numbers restricted to the scale $0, 1, 2, \ldots, i$ is the coefficient of t^x in the Quotient $\frac{(1-t^{i+1})(1-t^{i+2})\ldots\ldots(1-t^{i+j})}{(1-t)(1-t^2)\ldots\ldots(1-t^j)}$, and consequently this Quotient is Entire and Omnipositive.

Prove that the quotient obtained by writing $1-t^{i+kj}$, where k is any positive integer, in place of $1-t^{i+j}$, as the final factor in the numerator (leaving everything else unaltered), is also Entire and Omnipositive.

[Prof. Sylvester remarks that this is not a *fancy* question, but one that presented itself in a very important research in the Theory of Invariants.]

Solution by W. J. Curran Sharp, M.A.

If $\frac{(1-t^{i+1})(1-t^{i+2})\ldots\ldots(1-t^{i+j})}{(1-t)(1-t^2)\ldots\ldots(1-t^j)}$ be denoted by u_j,

$$\frac{(1-t^{i+1})(1-t^{i+2})\ldots\ldots(1-t^{i+j-1})(1-t^{i+kj})}{(1-t)(1-t^2)\ldots\ldots(1-t^{j-1})(1-t^j)}$$

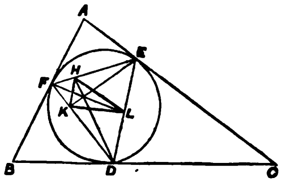
$$= u_{j-1}\frac{1-t^{i+kj}}{1-t^j} = u_{j-1}\frac{1-t^{i+j}+t^{i+j}-t^{i+kj}}{1-t^j} = u_j + u_{j-1}t^{i+j}\frac{1-t^{j(k-1)}}{1-t},$$

which proves what is required, as u_j, u_{j-1}, and $\frac{1-t^{j(k-1)}}{1-t^j}$ are all three Entire and Omnipositive.

6130. (By the Editor.)—Let ABC be a triangle; D, E, F the points where its sides are touched by its inscribed circle; H, K, L the feet of the perpendiculars from the vertices of the triangle DEF on the opposite sides; Δ the area of the triangle ABC, and R, r the radii of its circum-

scribed and inscribed circles; Δ_1 the area of the triangle DEF, and Δ_2 the area of the pedal triangle HKL; then prove that $\Delta, \Delta_1, \Delta_2$ are in continued proportion, whereof the common ratio is $\frac{r}{2R}$.

I. *Solution by* C. MORGAN, B.A.; Prof. GENESE, M.A.; *and others.*



The angles of the triangle DEF, and of its pedal triangle HKL, are, obviously,

$$\angle D = \tfrac{1}{2}(\pi - A), \text{ \&c. \&c.};$$
$$\angle H = \pi - 2D = A, \text{ \&c. \&c.};$$

also, the radius of the circle around HKL, which is the nine-point circle of the triangle DEF, is $\frac{1}{2}r$. Thus, since the triangles ABC, HKL are *similar*, and have their *linear* dimensions proportional to the radii of their circumscribed circles, we have $\Delta^{\frac{1}{2}} : \Delta_2^{\frac{1}{2}} = R : \frac{1}{2}r = 2R : r$.

[The sides of the triangle DEF are plainly parallel to the sides of its escribed triangle $O_1O_2O_3$; also ABC is the pedal triangle of the triangle $O_1O_2O_3$; hence the triangle ABC bears the same relation to the triangle $O_1O_2O_3$ as the triangle HKL does to the similar triangle DEF; hence we have $\Delta : \Delta_2 =$ triangle $O_1O_2O_3$: triangle DEF

$= $ circle $C_1O_2O_3$: circle DEF $= (2R)^2 : r^2$.

We thus see at once that the triangles HKL, DEF, ABC, $O_1O_2O_3$ are in continued proportion, the common ratio being as stated in the Question.]

II. *Solution by* Prof. COCHEZ; A. MCMURCHY, B.A.; *and others.*

The triangle HKL is similar to ABC; hence, we have

$$BC = 2R \sin A, \qquad EF = 2r \sin D = 2r \cos \tfrac{1}{2}A,$$
$$KL = EF \cos D = EF \sin \tfrac{1}{2}A = r \sin A;$$

therefore we have $\Delta^{\frac{1}{2}} : \Delta_2^{\frac{1}{2}} = 2R : r$.

Again, $\Delta_1 = \frac{1}{2}DE \cdot DF \cdot \sin D = 2r^2 \sin D \sin E \sin F$

$$= 2r^2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = \frac{2r^2 s \Delta}{abc} = \frac{r}{2R} \cdot \Delta;$$

hence $\Delta, \Delta_1, \Delta_2$ are in the continued proportion specified.

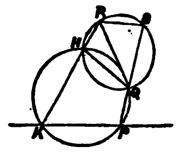
5648. (By J. L. MCKENZIE, B.A.)—P and Q are two given points, P on a given straight line, and Q on a given circle; show that it is possible to find another point R on the given circle, such that, if any straight line be drawn through R cutting the given circle in H and the given straight line in K, the points H, K, P, Q shall lie on a circle.

Solution by the EDITOR.

Let PQ meet the given circle in a second point S; then the points H, K, P, Q will lie on a circle, provided that the angle KPQ = QHR,

or $\angle KPQ + RSQ = QHR + RSQ = 180°$;

whence SR is parallel to PK, and the construction is obvious.



6133. (By H. McColl, B.A.)—It has been observed (1) that, whenever the events A, B, X happen together, the events C, D, E also happen; (2) that whenever the events B, C, Y happen together, the events D and E also happen; (3) that whenever the event C or D happens or E fails to happen, then B or X happens or A fails to happen, and also C or Y happens or B fails to happen; (4) that whenever X happens and A fails to happen, then Y happens and B fails to happen; and, conversely, whenever Y happens and B fails to happen, then X happens and A fails to happen. When may we conclude, from the occurrence or non-occurrence of the events A, B, C, D, E (without any reference to Y)—(1) that the event X will happen (or has happened); (2) that the event X will *not* happen (or has not happened)?

I. *Solution by the* PROPOSER.

The premises of the problem are as follows:—

$$abx : cde,\ bcy : de,\ c+d+e' : (a'+b+x)(b'+c+y),\ a'x = b'y \ldots (1, 2, 3, 4).$$

From these premises, by rule 23 and formula 6 of my third paper in the *Proceedings* of the London Mathematical Society, we get

$$x'y : (b'+c'+de)(c'd'e+a'+b)\,b,\quad x'y' : c'd'e+(a'+b)(b'+c);$$

that is, $\quad x'y : (c'+de)\,b,\quad x'y' : c'd'e+a'b'+a'c+bc,$

for, from formula 2, $(c'd'e+a'+b)\,b = (c'd'e+a'+1)\,b = b.$

Hence, adding antecedent to antecedent, and consequent to consequent, we have $\quad x' : b+bde+c'd'e+a'b'+a'c,$

for $\quad x'(y+y') = x',\ \text{ and }\ b\,(c+c') = b.$

Omitting the redundant term bde, and observing that $b+a'b' = b+a'$, this becomes $\quad x' : b+a'+a'c+c'd'e.$

Omitting the redundant term $a'c$, and transposing, we have

$$ab'(c+d+e') : x.$$

Translated into ordinary language, this may be read thus:—When we know that the event A has happened, and that B has not happened, and also that either C or D has happened or that E has not happened, then we may infer that X will happen or has happened.

Again, from the premises of the problem, we get

$$xy : (a'+b'+cde)(b'+c'+de)(ab+a'b'),\quad xy' : (a'+b'+cde)(c'd'e+b'+c)\,a.$$

Using formula 5 of the before-cited paper, and omitting the term a' in the first bracket of the second consequent, because it is inconsistent with the last factor a, these two implications become

$$xy : (b' + a'c' + a'de + cde)(ab + a'b'), \quad xy' : (b' + cde)\, a.$$

The term $a'de$ in the first bracket of the first consequent is redundant (see rule 20), for, by rule 22,

$$a'de\,(b' + a'c' + cde)' = a'de\,(b' + c' + e)' = 0.$$

Omitting this redundant term, then multiplying the factors, and adding antecedent to antecedent, and consequent to consequent, we get

$$x : ab' + acde + \underline{abcde} + a'b'.$$

Omitting the redundant term underlined, and observing that

$$ab' + a'b' = (a + a')\, b' = b',$$

this becomes $x : b' + acde$, or, transposing, $b\,(a' + c' + d' + e') : x'$.

Translated into ordinary language, this may be read as follows:—When we know that the event B has happened, and that one (or more) of the events A, C, D, E has *not* happened, then we may infer that the event X will not happen, or has not happened.

[Professor JEVONS's method of solving logical problems—adopted by Professor TANNER in his solution of Quest. 5861 (*Reprint*, Vol. XXXI., p. 43)—is simple in theory, and when (as in Quest. 5861) only 4 or 5 elementary constituents have to be taken into account, it is not very tedious in practice; but the difficulty of applying his method—as he frankly points out himself in his *Elementary Lessons in Logic*—increases rapidly with the number of elementary constituents, and when these are 7 or 8 in number the method is thought to become altogether impracticable.]

II. *Solution by* C. J. MONRO, M.A.

Put H for cde, and K for what Mr. McCOLL writes $c + d + e'$; then, in BOOLE's notation, only with accents to negative symbols,

$$abH'x = 0, \quad b\,(c - H)\, y = 0, \quad K\,(ab'x' + bc'y') = 0,$$

$$a'x\,(1 - b'y) + (1 - a'x)\, b'y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1\text{—}4).$$

The left sides consist of positive class terms, and may be combined by addition.

Add (2), (3), (4), making y successively 1 and 0. This gives

$$b\,(c - H) + Kab'x' + a'bx + (1 - a'x)\, b' \quad \text{and} \quad Kbc' + Kab'x' + a'x.$$

The product of these gives the result of eliminating y, which, therefore (since $c'H = 0$), is $\quad a'b\,(c + H')\, x + Kab'x' = 0.$

Add (1); the whole system, irrespectively of y, is then reduced to

$$(a'bc + bH')\, x + Kab'x' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

Hence $$x = \frac{Kab'}{Kab' - a'bc - bH'}, \quad = \frac{0}{-H'}\, ab + \frac{K}{K}\, ab' + \frac{0}{-c - H'}\, a'b + \frac{0}{0}\, a'b';$$

which, since $c + H' = 1 + c\,(1 - de)$ and cannot vanish,

$$= \tfrac{0}{0}\, abH + 0\, abH' + ab'K + \tfrac{0}{0}\, ab'K' + 0\, a'b + \tfrac{0}{0}\, a'b';$$

that is to say (since $K' = c'd'e$), that, irrespectively of Y, X always happens when A happens without B, but with C or with D or without E

Remarks by J. J. WALKER, M.A.

Supposing two such films could form the surfaces bounding two solids, connected along the perimeters of the given circles, and convex outwardly from symmetry, it is apparent that either part would be a surface of revolution round the line joining the centres as axis; and the question would be reduced to "finding the curve of given length such that the surface generated by its revolution round a given line may be given, and that the volume of the generated solid may be a maximum." These conditions are too many. For the solutions of the problem resulting from omitting either of the first two conditions, reference may be made to Professor JELLETT's *Calculus of Variations*, pp. 160, 161. It does not seem possible to form a cushion having a continuous surface from two such films, connected as in the question, unless a *corrugated* shape were admissible and practicable.

[Col. CLARKE states that the problem is to determine the law of the corrugations or creasings (infinitely small); that he believes the law in question to be simple, and that he once investigated it, and, in illustration thereof, folded some paper, of which he sends a specimen.]

3753. (By W. H. H. HUDSON, M.A.)—There are three towns A, B, C, forming a railway triangle; the population of each is proportional to the line joining the other two. Assuming that the number of trains in and out of each varies as its population, prove that the number of train-miles run between A and B varies as $\cos^2 \frac{1}{2}C$. Show also that, for a given aggregate population and length of line, the total number of train-miles is greatest when the towns are equidistant.

I. *Solution by* W. SIVERLY; J. A. KEALY, M.A.; *and others.*

Let x, y, z be the number of trains between (A, B), (A, C), (B, C), and λ some constant; then we have $x+y = \lambda a$, $x+z = \lambda b$, $y+z = \lambda c$. Putting, as usual, $\frac{1}{2}(a+b+c) = s$, and $s-a = s_1$, &c., we have $x = \lambda s_1$, $y = \lambda s_2$, $z = \lambda s_3$. Assume $\lambda = \dfrac{s}{abc}\lambda'$, then $x = \dfrac{\lambda' ss_3}{abc}$, $y = \dfrac{\lambda' ss_2}{abc}$, $z = \dfrac{\lambda' ss_1}{abc}$;

hence the train-miles between A and B $= \dfrac{\lambda_1 ss_3}{ab}$ varies as $\cos^2 \frac{1}{2}C$, between A and C varies as $\cos^2 \frac{1}{2}B$, and between B and C as $\cos^2 \frac{1}{2}A$. Also $A+B+C = \pi$, $u = \cos^2 \frac{1}{2}A + \cos^2 \frac{1}{2}B + \cos^2 \frac{1}{2}C = \max.$ Differentiating with respect to A and B successively, we have

$$1 + \frac{dC}{dA} = 0, \quad 1 + \frac{dC}{dB} = 0;$$

$$\frac{du}{dA} = -\tfrac{1}{2}\sin A - \tfrac{1}{2}\sin C \frac{dC}{dA}, \quad \frac{du}{dB} = -\tfrac{1}{2}\sin B - \tfrac{1}{2}\sin C \frac{dC}{dB};$$

whence, replacing $\dfrac{dC}{dA}$, $\dfrac{dC}{dB}$ by their values, we obtain

$$\tfrac{1}{2}\sin A = \tfrac{1}{2}\sin B = \tfrac{1}{2}\sin C, \text{ therefore } A = B = C = \tfrac{1}{3}\pi.$$

but $y=0$ when $x=0$ and when $x=a$, hence we have

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{Y}(x^3-a^2x)-\tfrac{1}{24}w(x^4-a^3x);$$

and, at B, we have $$\mathrm{EI}\frac{dy}{dx} = \tfrac{1}{3}\mathrm{Y}a^2-\tfrac{1}{8}wa^3 \text{ .... (1).}$$

For the segment BO, take B as origin and BO as axis of x, then we have

$$\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{X}x+\mathrm{Y}(x+a)-\tfrac{1}{2}w(x+a)^2;$$

but $y=0$ when $x=0$ and when $x=a$, hence we have

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{X}(x^3-a^2x)+\tfrac{1}{6}\mathrm{Y}(x^3+3ax^2-4a^2x)-\tfrac{1}{24}w(x^4+4a^3x+6a^2x^2-11a^3x);$$

therefore, at B and O, respectively, we have

$$\mathrm{EI}\frac{dy}{dx} = -\frac{a^2}{6}\mathrm{X}-\frac{4a^2}{6}\mathrm{Y}+\tfrac{11}{24}a^3w \text{ ....(2),}$$

$$\mathrm{EI}\frac{dy}{dx} = \frac{a^2}{3}\mathrm{X}+\frac{5a^2}{6}\mathrm{Y}-\tfrac{17}{24}a^3w \text{ ....(3).}$$

But expressions (1) and (2) must be equal, and by symmetry we see that $\frac{dy}{dx}$ vanishes at the point O; hence we obtain, finally,

$$4\mathrm{X}+24\mathrm{Y} = 14aw, \quad 8\mathrm{X}+20\mathrm{Y} = 17aw,$$

$$\mathrm{X} = \frac{8aw}{7} = \tfrac{4}{7} \text{ weight of the rod.}$$

II. *Solution by the* Proposer.

Denoting by a the semi-length of the rod, by W its semi-weight, and by W′ the fraction of W supported by each prop in question, and supposing the latter removed and replaced each by a force equal and opposite to W′. Then, since, on known principles, the depression y, at any point of the rod distant x from its centre, due to the action of gravity, is given by the formula

$$\mathrm{EI}\,.\,y = \tfrac{1}{48}\,.\,\frac{\mathrm{W}}{a}\,.\,x\,.\,(a-x)\,.\,(2a^2-3ax+2x^2);$$

and the elevation y', at any point of it, distant x' from its centre, due to the action of the upward force W′, by the formula

$$\mathrm{EI}\,.\,y' = \tfrac{1}{12}\,.\,\frac{\mathrm{W}}{a^3}\,.\,x'^2\,.\,(a-x')^2\,.\,(a^2+4ax'-x'^2),$$

where E and I have their ordinary well-known significations; hence, in order that y' should neutralise y at the point for which $x'=x=\frac{1}{2}a$, we must have $$\tfrac{11}{96}\mathrm{W}' = \tfrac{1}{24}\mathrm{W}, \quad \text{or} \quad \mathrm{W}' = \tfrac{4}{11}\mathrm{W},$$
which accordingly is the fraction required.

2432. (Proposed by Colonel Clarke, C.B., F.R.S.)—Two perfectly flexible and inextensible films, being equal circles in form, are fastened together along the perimeter; what is the greatest amount of air that this could be made to hold (as an *air-cushion*)?

and if x_1, x_2, x_3, x_4 be the values of x for the four chords, we shall have the four equations,

$$A\Delta_1^2 + Bx_1^2 + Cx_1 + D = 0, \text{ \&c.} = 0, \text{ \&c.} = 0, \text{ \&c.} = 0; \text{ whence}$$

$$\begin{vmatrix} \Delta_1^2, & x_1^2, & x_1, & 1 \\ \Delta_2^2, & x_2^2, & x_2, & 1 \\ \Delta_3^2, & x_3^2, & x_3, & 1 \\ \Delta_4^2, & x_4^2, & x_4, & 1 \end{vmatrix} = 0, \text{ or } \Delta_1^2 \begin{vmatrix} x_2^2, & x_2, & 1 \\ x_3^2, & x_3, & 1 \\ x_4^2, & x_4, & 1 \end{vmatrix} - \Delta_2^2 \begin{vmatrix} x_1^2, & x_1, & 1 \\ x_3^2, & x_3, & 1 \\ x_4^2, & x_4, & 1 \end{vmatrix} + \text{\&c.} = 0,$$

or $\quad \Delta_1^2 (x_2-x_3)(x_2-x_4)(x_3-x_4) - \Delta_2^2(\ldots) + \Delta_3^2(\ldots) - \Delta_4^2(\ldots) = 0,$

or $\quad (2.3)(2.4)(3.4)\Delta_1^2 - (1.3)(1.4)(3.4)\Delta_2^2$
$$+ (1.2)(1.4)(2.4)\Delta_3^2 - (1.2)(1.3)(2.3)\Delta_4^2 = 0.$$

6134. (By T. Cotterill, M.A.)—Let aa', bb', cc' be the opposite vertices of four co-planar lines: then, if three systems of conics pass respectively through $(bb'cc')$, $(cc'aa')$, $(aa'bb')$; prove (1) that the invariant $\Theta\Theta' - \Delta\Delta'$ of two conics, one from each system, vanishes, and the double chord of intersection is harmonic to the third system of conics; and (2) give the reciprocal of these properties.

I. *Solution by* J. J. Walker, M.A.

1. Suppose aa', bb' to meet in o, and let cc', $c'o$, oc be x, y, z respectively; then, without any loss of generality, bc', ac' may be taken as $x+y$, $x-y$, and $b'c$, ac as $x+z$, $x-z$. The three systems of conics will then be

$$S_1 \equiv x^2 + yz + kzx + kxy, \quad S_2 \equiv x^2 - yz + k'zx - k'xy,$$
$$S_3 \equiv (1-k'')x^2 + k''y^2 - z^2.$$

For the system S_3S_2, $\quad \Delta = (k''-1)k'', \quad \Delta' = 2(k'^2-1),$

$$\Theta = -2k'', \quad \Theta' = (1-k'')(k'^2-1), \text{ so that } \Theta\Theta' - \Delta\Delta' \equiv 0.$$

Again, bb', the common chord of the systems S_3, S_1, is $y+z$, and the pole of this line with respect to S_2 is determined by

$$2x' - k'y' + k'z' = 0, \quad k'x' + z' = y' - k'x',$$

a point on $y-z=0$, which is aa', the chord common to the systems S_3, S_2.

2. Let $\Delta_1, \Delta_1', \Theta_1, \Theta_1'$ refer to the conics reciprocal to the general conics $ax^2 + \ldots$, $a'x^2 + \ldots$, then it is readily verified that (to a factor *près*)

$$\Delta_1 = \Delta^2, \quad \Delta_1' = \Delta'^2, \quad \Theta_1 = \Delta\Theta', \quad \Theta_1' = \Delta'\Theta;$$

so that $\quad \Theta_1\Theta_1' - \Delta_1\Delta_1' = \Delta\Delta'(\Theta\Theta' - \Delta\Delta').$

Hence, if AA′, BB′, CC′ are three pairs of straight lines joining four co-planar points, and if three systems of conics touch respectively (BB′CC′), (CC′AA′), (AA′BB′), the invariant $\Theta\Theta' - \Delta\Delta'$ of two conics, one from either of two of the systems, vanishes; and the polar of the point of intersection of the pair of tangents common to two systems, with respect to any one of the third system passes through the intersection of the tangents common to that third system and either of the former two.

II. *Solution by* Professor Wolstenholme, M.A.

1. Let the four lines be $x \pm y \pm z = 0$; and let $(0, 1, 1)$, $(0, 1, -1)$, $(1, 0, 1)$, $(-1, 0, 1)$, $(1, 1, 0)$, $(1, -1, 0)$ be the coordinates of a, a', b, b', c, c'; then the equation of a conic through $aa'bb'$ will be

$$x^2 + y^2 - z^2 + 2hxy = 0 \equiv S,$$

and that of a conic through $aa'cc'$ will be

$$x^2 - y^2 + z^2 + 2gzx = 0 \equiv S',$$

and the discriminant of $kS + S'$ is

$$(k+1)(k-1)^2 + (k-1)g^2 - (k-1)k^2h^2;$$

so that $\quad \Delta = 1 - h^2, \quad \Theta = h^2 - 1, \quad \Theta' = g^2 - 1, \quad \Delta' = 1 - g^2,$

and the true invariant relation $\Theta\Theta' = \Delta\Delta'$ is always satisfied.

The third system of conics is

$$-x^2 + y^2 + z^2 + 2fyz = 0,$$

and the common chord of the two former is $\; x = 0, \; y^2 = z^2,$

and the range on $x = 0$, formed by $y^2 = z^2$, $y^2 + z^2 + 2fyz = 0$, is harmonic.

Otherwise:—if a, a' be projected into circular points, bb', cc' will be two pairs of anti-points, or we may take b, b' for the real, and c, c' for the impossible foci of a conic, say $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; then a conic through $aa'bb'$ will be a circle $\qquad x^2 + y^2 - 2ky = a^2 - b^2,$

and a conic through $aa'cc'$ will be a circle

$$x^2 + y^2 - 2hx + a^2 - b^2 = 0;$$

and these circles cut at right angles; and, in that case, the harmonic envelope degenerates into a point-pair (the centres of the circles), and the relation $\Theta\Theta' = \Delta\Delta'$ is satisfied. The conics through $bb'cc'$ are the system of rectangular hyperbolas $x^2 - y^2 + 2fxy = a^2 - b^2$, and divide the segment whose extremities are the circular points aa' harmonically.

2. If aa', bb', cc' be the three pairs of straight lines through the four points of a quadrangle, and two systems of conics be drawn touching $aa'bb'$ and $aa'cc'$ respectively, the invariant relation $\Theta\Theta' = \Delta\Delta'$ will be satisfied between any two conics, one from each system, and the two common tangents a, a' will be harmonically conjugate to any conic touching $bb'cc'$. The invariant relation is unaltered, since it is the condition both that the harmonic locus degenerate into a line-pair, and that the harmonic envelope degenerate into a point-pair.

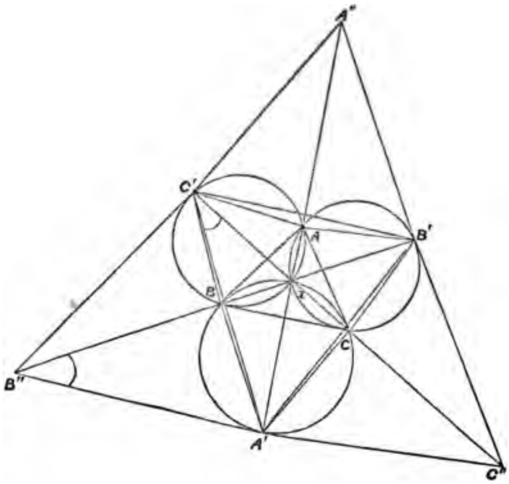
5782. (By T. Mitcheson, B.A., L.C.P.)—If equilateral triangles be drawn on the sides of a triangle, prove that the lines that join the vertices of these triangles with the opposite corners of the original triangle, and

the circles drawn round these triangles, meet at a point where the sides of the triangle subtend equal angles.

5932. (By Professor COCHEZ.) — Connaissons les sommets de trois triangles équilatéraux construits sur les côtés d'un triangle, construire ce triangle.

Solution by Professor COCHEZ; H. N. CAPEL, LL.B.; *and others.*

(Quest. 5782.) Soit ABC un triangle quelconque; BCA′, CAB′, ABC′ les triangles équilatéraux construits sur les côtés de ce triangle.



Les triangles ACC′, BAB′ sont égaux comme ayant un angle C′AC = B′AB compris entre côtés égaux, donc CC′ = BB′; on démontrerait, de même que BB′ = AA′. Donc AA′ = BB′ = CC′.

Circonscrivons des cercles aux triangles BCA′, BAC′. Soit I leur second point d'intersection. Si l'on joint AI, BI, CI, les angles BIC et AIB sont égaux à 120°; il en sera de même de l'angle AIC. Le cercle circonscrit à AB′C passera également en I.

Menons IA′, IB′, IC′. Les angles consécutifs A′IC, CIB′, AIB′ valent 60°. Donc IA′, IB′, IC′ sont les prolongements de AI, BI, CI. Donc les trois droites et les trois cercles se coupent en un même point.

De ce point I on voit également sous un angle de 120° les côtés du triangle A′B′C′. Si on effectuait la même construction sur les côtés du triangle A′B′C′, les droites A′A″, B′B″, C′C″ sont aussi égales et se coupent en un certain point duquel on voit les côtés du triangle A′B′C′ et ceux de ABC sous un angle de 120°. Ce point coincide donc avec I. Donc les droites AA′A″, BB′B″, CC′C″ sont concourrantes.

(Quest. 5932.) Ce théorème permet de résoudre le problème dans la Quest. 5932 [which is, in fact, identical with Quest. 5811, whereof a solution is given on p. 60 of this volume of the *Reprint*].

Soient A′, B′, C′ les points donnés, joignons ces points et sur les droites

A′B, C′B′, A′C′ construisons des triangles équilatéraux. Joignons A″A′, B″B′, C″C′. Ces droites se coupent en I. Si on prend les milieux A, B, C de ces droites, et si on les joint, on a le triangle cherché ABC.

En effet, les triangles A′BB″, A′CC′ sont égaux, car CC′ = BB″ comme moitiés de droites égales. A′C′ = A′B″ par construction, et le quadrilatère A′IC′B″ étant inscriptible puisque l'angle en B″ vaut 60° et l'angle C IA′ 120°, donc les angles A′C′C et A′B″B sont égaux, donc AB′ = A′C. Mais B″A′B = 60°, il en est dès lors de même de BA′C. Donc BA′C est le triangle équilatéral construit sur BC. On démontrerait de même que AB′C et BAC′ sont équilatéraux. Donc ABC est bien le triangle cherché.

6060. (By J. YOUNG, B.A.)—Prove that any one of the three following equations is derivable from the other two :—

$$\begin{aligned} A &\equiv a^2(x^2+xy+y^2)-axy(x+y)+x^2y^2=0, \\ B &\equiv a^2(y^2+yz+z^2)-ayz(y+z)+y^2z^2=0, \\ C &\equiv a^2(z^2+zx+x^2)-azx(z+x)+z^2x^2=0. \end{aligned}$$

Solution by J. O'REGAN; I. H. TURRILL, M.A.; *and others.*

We have $\dfrac{z^2A-x^2B}{ay(z-x)}=a(xy+yz+zx)-xyz=\dfrac{x^2B-y^2C}{az(x-y)}$;

therefore, if any two of A, B, C vanish, so likewise must the third.

5883. (By Prof. TANNER, M.A.)—Given k, a positive proper fraction, and the series $k'_1, k'_2, \dots k_1, k_2 \dots$ formed from it by successive application of the equations $(1+k'_n)(1+k_{n-1})=2,\ \ k_n^2+k_n'^2=1$; show that, when n is indefinitely increased, $(1+k'_n)^{2^n-1}$ tends to the limit 1.

Solution by the PROPOSER.

To prove this it is only necessary to show that, when n is indefinitely increased, $2^{n-1}k'_n$ tends to vanish. [Now, from the given equations, we have

$$k'^2_{n-1}=\frac{4k'_n}{(1+k'_n)^2} \quad\text{or}\quad k'_n=\frac{(1+k'_n)^2}{4}\cdot k'^2_{n-1}.$$

Thus k'_n is positive. Also, since k'_n is less than 1, the coefficient of k'^2_{n-1} is less than 1; therefore $\quad k'_n<k'^2_{n-1}.$

Hence we have in succession

$$0<k'_n<k'^2_{n-1}<k'^4_{n-2}<\dots<k'^{2^n};\ \text{therefore}\ 0<2^nk'_n<2^nk'^{2^n}.$$

Now the right-hand expression lends to the limit 0 as n is indefinitely increased (for, k' being less than 1, mk'^m becomes 0 when $m = \infty$). Hence the same must be true of $2^n k'_n$, and the theorem is proved.

[In DURÈGE's *Elliptic Functions*, p. 212, 2nd ed., the author seems to assume that a^m tends to the value 1 if a does so. But, from the above, this is evidently insufficient; we require also that $(a-1)m$ should vanish, and this is not always the case when m is infinite. For example, we may cite the well-known instance, $(1+kx)^{x^{-1}}$, which, when x becomes infinitesimal, tends to the limit e^k, although $1+kx$ tends to the limit 1.]

6167. (By Dr. HOPKINSON, F.R.S.)—Two magneto-electric machines, with permanent inducing magnets, are used for the transmission of power, the first machine driven by an engine with uniform velocity, and producing a current which is passed through the second machine, the second machine acting as a magneto-electric engine. When the second machine is standing, but the current of the first is passed through its coil, the first machine is found to absorb a power W from the engine. Neglecting the loss due to friction, to disruptive discharges, and to short circuiting of the coils; prove (1) that the maximum power obtainable from the second machine is $\frac{1}{4}$W, and (2) that then the first machine absorbs $\frac{1}{2}$W from the engine.

Solution by Professor G. CAREY FOSTER, F.R.S.

Let the work taken in by the first machine be, when the second is standing, $= W_1$, and when the second is running, $= W'_1$; the work given out by the second machine $= W_2$; the electromotive force of first machine, when the second is standing, $= E_1$, and when the second is running, $= E'_1$; the inverse electromotive force of second machine $= E_2$; and the resistance of the circuit $= R$. Among these quantities we have, by JOULE's law of the energy of electric currents, the equations

$$W_1 = \frac{E_1^2}{R}, \qquad W'_1 = \frac{E'_1(E'_1 - E_2)}{R}, \qquad W_2 = \frac{(E'_1 - E_2)E_2}{R}.$$

The two factors in the numerator of the last expression have the constant sum E'_1; therefore W_2 has a maximum value when they are both equal to $\frac{1}{2}E'_1$. Hence the maximum of W_2 is always equal to $\frac{1}{2}W'_1$, as stated in the problem; but to determine the relation between W'_1 and W_1, a further condition is required. We may consider three special cases:—(1) When the speed of the first machine is the same, whether the second is standing or running. (This condition might be practically realised by help of a governor, and might be nearly realised without one, if the work taken in by the magneto-electric machine were a small fraction of that given out by the prime mover.) (2) The work taken in by the first machine is constant, or $W_1 = W'_1$. (This would be approximately the case when there is no governor, and the first machine takes in nearly all the useful work given out by the prime mover.) (3) The strength of the current is constant, or $E_1 = E'_1 - E_2$.

Case 1.—Here $E_1 = E'_1$, and therefore, when W_2 is a maximum, $E_2 = \frac{1}{2}E_1$, and therefore $W'_1 = \frac{1}{2}W_1$. This is evidently the case contemplated in the problem.

Case 2.—Since $W_1 = W'_1$, the maximum of W_2 becomes $= \frac{1}{2}W_1$. In this case we have $E_2 = \frac{E_1}{\sqrt{2}}$, since $E_2 = \frac{1}{2}E'_1$, and $E_1^2 = E'_1(E'_1 - E_2)$.

Case 3.—Here, since $E'_1 - E_2 = E_2$ as the condition for the maximum of W_2, and also by supposition $= E_1$, we have $W_2 = W_1$, and $W'_1 = 2W_1$.

5494. (By Professor Tait.)—Determine the number of *distinct* knots with 8 crossings, and figure one form of each.

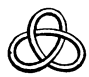
Remarks by the Proposer.

In a paper which appeared in the *Transactions* of the Royal Society of Edinburgh (1877), I have shown *how* this can be done. I had leisure to solve it completely for 3, 4, 5, 6, 7 crossings, but for the 8-fold knot (at least by *my* process) over 4000 separate cases have to be investigated, and I cannot spare the requisite time. I have shown that the number of knots (distinct from one another) with these numbers of crossings, is as follows:—

Number of crossings... 3, 4, 5, 6, 7, 8, ...
Number of species ... 1, 1, 2, 4, 8, ..., ...

It is not likely that the geometric series will hold for another term, but I am desirous of knowing what the number for 8 is. When 9 is in question, the selection has to be made from among over 40,000!

Here is one 8-fold knot as a specimen. It is peculiar, inasmuch as it is what I call *amphicheiral*, that is to say, it can (if made of cord) be changed into its own perversion or image in a plane mirror. This property the annexed *trefoil* does not possess.



6156. (By Professor Townsend, F.R.S.)—The axis of principal moment of any system of forces in a common space being supposed taken with respect to a variable point on any fixed right line in the space; show that it generates a skew paraboloid, whose axis of figure is the common perpendicular to the directive right line, and to the central axis of the system.

Solution by J. J. WALKER, M.A.

Let OX ($=a$), the common perpendicular to the directive line OL and the central axis, OZ parallel to the latter, and OY perpendicular to OX, OZ, be taken as axes of coordinates; and let the equation to OL be $qy-z=0$. Then, if R, pR are the magnitudes of the resultant force and of the minimum principal moment respectively; the axis of principal moment relative to a point $(0, y', z')$ on OL will be the line joining that point with $x''=\text{R}y'$, $y''=y'+a\text{R}$, $z''=z'+p\text{R}$, and its equation will therefore be $x:\text{R}y'=y-y':a\text{R}=z-qy':p\text{R}$. Eliminating y', we have $(aq-p)^2x=(qy-z)(az-py)$, a hyperbolic paraboloid, having OX as its axis, for the surface containing the axis of principal moment at points on OL. [Mr. WALKER remarks that this theorem formed the second part of Prof. MINCHIN's Quest. 5566, of which a Quaternion proof by him is given on p. 40 of Vol. XXXII. of our *Reprints*.]

5762. (By Professor SYLVESTER, F.R.S.)—If $f(x, y)$ is a quantic of n dimensions in x, y, prove that $(n-1)\left(\dfrac{df}{dx}\right)^2-nf\dfrac{d^2f}{dx^2}$ is a negative numerical multiplier of y^2 into the Hessian of f.

6077. (By W. J. C. SHARP, M.A.)—If any equation have three equal roots, prove that (1) the Hessian will have the same root as a double root; or generally, (2) that if any equation have a p-ple root, the same quantity is a $2(p-1)$-ple root of the Hessian.

Solution of both Questions by W. J. CURRAN SHARP, M.A.

By the known properties of homogeneous functions, we have

$$n(n-1)f=x^2\frac{d^2f}{dx^2}+2xy\frac{d^2f}{dx\,dy}+y^2\frac{d^2f}{dy^2},\text{ and }(n-1)\frac{df}{dx}=x\frac{d^2f}{dx^2}+y\frac{d^2f}{dx\,dy};$$

therefore $(n-1)^2\left(\dfrac{df}{dx}\right)^2-n(n-1)f\dfrac{d^2f}{dx^2}$

$$=\left(x\frac{d^2f}{dx^2}+y\frac{d^2f}{dx\,dy}\right)^2-\frac{d^2f}{dx^2}\left(x^2\frac{d^2f}{dx^2}+2xy\frac{d^2f}{dx\,dy}+y^2\frac{d^2f}{dy^2}\right)$$

$$=-y^2\left\{\frac{d^2f}{dx^2}\frac{d^2f}{dy^2}-\left(\frac{d^2f}{dx\,dy}\right)^2\right\},$$

therefore $$(n-1)\left(\frac{df}{dx}\right)^2-nf\frac{d^2f}{dx^2}=-\frac{y^2}{n-1}\text{H}.$$

From this it follows that, if the equation $f=0$ have p roots each $=a$, the Hessian H will have $2(p-1)$ roots each $=a$.

If, then, the equation have $(n-1)$ roots each $=a$, the Hessian will be $(ac-b^2)(x-a)^{2(n-2)}$; by comparing which with ordinary form, the requisite conditions may be obtained. Thus, the conditions that quartic $ax^4+4bx^3+\&c.=0$ may have three equal roots, are

$$\frac{ad-bc}{2(ac-b^2)}=\frac{ae+2bd-3c^2}{3(ad-bc)}=\frac{3(be-cd)}{ae+2bd-3c^2}=\frac{2(ce-d^2)}{be-cd},$$

which give $T=0$ and $S=0$, the necessary and sufficient conditions, as the above equations are not all independent.

[Prof. SYLVESTER remarks that the effect of the theorem in Quest. 5762 is to show that, in his extension of NEWTON'S Rule, the double series (of which the double permanences are to be counted in order to determine a superior limit to the number of real roots included between any two given quantities) may be written as follows:

$$fx,\ f'x,\ f''x,\ \ldots\ldots,\ f'^{(n-1)}x,\ f'^{(n)}x;$$
$$-1,\ Hfx,\ Hf'x,\ \ldots\ldots,\ Hf'^{(n-2)}x,\ -1;$$

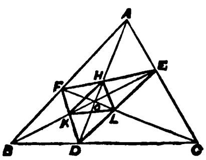
as, for example, for a biquadratic the series may be taken as

$$fx,\ f'x,\ f''x,\ f'''x,\ f''''x;$$
$$-1,\ Hfx,\ Hf'x,\ Hf''x,\ -1;$$

where $H\phi(x)$, in general, means the Hessian of ϕx, regarded as a homogeneous function of x and 1. Since the $(n-2)^{\text{th}}$ derivative of f is a quadratic in x, it follows that the last of the H's is always a constant. This the PROPOSER believes to be a most important transformation of the theorem as it has been hitherto given.]

4208. (By Professor MONCK, M.A.)—Three lines AD, BE, CF are drawn from the vertices of a triangle ABC through a point O to the opposite sides; prove that $2\Delta ABC : \Delta DEF = AO \cdot BO \cdot CO : DO \cdot EO \cdot FO$.

Solution by R. TUCKER, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*



$$\frac{AO}{DO} = \frac{\Delta AOC}{\Delta DOC} = \frac{\Delta AOC}{\Delta BOC}\cdot\frac{\Delta BOC}{\Delta DOC} = \frac{AF}{BF}\cdot\frac{BC}{DC};$$

also $$\frac{BO}{EO} = \frac{AC\cdot BD}{AE\cdot CD},$$

and $$\frac{CO}{FO} = \frac{AB\cdot CE}{BF\cdot AE};$$

therefore $$\frac{AO\cdot BO\cdot CO}{DO\cdot EO\cdot FO} = \frac{AB\cdot BC\cdot CA\cdot AF\cdot BD\cdot CE}{(AE\cdot CD\cdot BF)^2} = \frac{AB\cdot BC\cdot CA}{AE\cdot CD\cdot BF}$$

$$= \text{(by Quest. 4184, *Reprint*, Vol. XX., p. 104)}\ \frac{abc}{2R\Delta DEF} = \frac{2\Delta ABC}{\Delta DEF}.$$

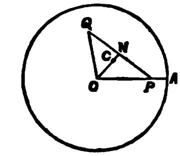
[Let AD, BE, CF meet EF, FD, DE in H, K, L; and let Δ be the area of the primitive triangle ABC, and Δ_2 the area of the second derivative triangle HKL; then, putting

$$AO\cdot BO\cdot CO = p_1, \text{ and } DO\cdot CO\cdot EO = p_2,$$

it may be readily shown that $2^2\Delta : \Delta_2 = p_1 : p_2$; and proceeding, *seriatim*, to the nth derivative triangle, we shall have $2^n\Delta : \Delta_n = p_1 : p_2$. The foregoing solution furnishes also a solution of Quests. 5899, 6003.]

6132. (By Elizabeth Blackwood.)—If O be the centre of a given circle of radius r, P and Q random points in it, and C the centroid of the triangle OPQ; show that the average value of OC is $\dfrac{128r}{135\pi}$.

Solution by Prof. Seitz; A. Martin, M.A.; *and others.*



Let $OA = r$, $OP = x$, $OQ = y$, $OC = z$, $\angle POQ = \theta$. Then we have $z = \frac{1}{3}(x^2 + y^2 + 2xy\cos\theta)^{\frac{1}{2}}$; the limits of θ are 0 and π; of x, 0 and r; and of y, 0 and x. Hence the average value of OC is

$$\int_0^\pi\int_0^r\int_0^x z\,d\theta\,x\,dx\,y\,dy + \int_0^\pi\int_0^r\int_0^x d\theta\,x\,dx\,y\,dy$$

$$= \frac{8}{3\pi r^4}\int_0^\pi\int_0^r\int_0^x (x^2 + y^2 + 2xy\cos\theta)^{\frac{1}{2}}\,d\theta\,x\,dx\,y\,dy$$

$$= \frac{2}{9\pi r^4}\int_0^\pi\int_0^r \Big\{8\cos\tfrac{1}{2}\theta + 40\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 1 + 3\cos 2\theta - 6\sin^2\theta\cos\theta\log(1 + \sec\tfrac{1}{2}\theta)\Big\}\,d\theta\,x^4\,dx$$

$$= \frac{2r}{45\pi}\int_0^\pi \Big\{8\cos\tfrac{1}{2}\theta + 40\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 1 + 3\cos 2\theta - 6\sin^2\theta\cos\theta\log(1 + \sec\tfrac{1}{2}\theta)\Big\}\,d\theta = \frac{128r}{135\pi}.$$

6191. (By Professor Crofton, F.R.S.)—Show that the common centre of gravity of any two arcs AB, A′B′ of the same circle, lies in the diameter perpendicular to the line joining the middles of the chords AA′, BB′.

Solution by C. J. Monro, M.A.; Morgan Jenkins, M.A.; *and others.*

In other words, if a quadrilateral is inscribed in a circle, the bisector of the diagonals is perpendicular to the mass-vector of arcs cut off by opposite sides, the origin being taken at the centre. Let r be the radius; m the mass of unit arc; α, β, α', β' vectors of A, B, A′, B′ from the centre. Since $\int e^{i\theta}\,d\theta = -ie^{i\theta}$, the mass-vector of opposite arcs is

$$\pm\, mri\,(\alpha - \beta + \alpha' - \beta'),$$

perpendicular to $-\frac{1}{2}(\alpha + \alpha') + \frac{1}{2}(\beta + \beta')$ bisecting the diagonals.

6047. (By Donald McAlister, B.A., B.Sc.)—Prove the following rule for finding the centre of inertia of the surface of a right cone cut off

by an oblique plane:—Make a plane section parallel to the base and at two-thirds of its distance from the vertex; let the axis cut this plane in A, and let B be its centre; then C, the point required, lies in AB produced, and AB : AC = 2 : 3.

Solution by R. LEIDHOLD, M.A.; T. R. TERRY, M.A.; *and others.*

Let a be the vertex and b the centre of the base. Consider the surface as made up of an infinite number of small triangles, having a for vertex, and for bases the sides of the polygon, which ultimately becomes the ellipse whose centre is b. Thus, the centre of inertia must be in the plane two-thirds of the way from a to the base.

Next, take a plane perpendicular to the axis. All the surface is equally inclined to this plane; therefore the centre of inertia of the surface projects orthogonally into the centre of inertia of the area of the projection; that is, into the projection of b; but $a\mathrm{B} = \frac{2}{3}ab$; therefore $\mathrm{AB} = \frac{2}{3}\mathrm{AC}$.

3916. (By S. WATSON.)—If O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles of a triangle ABC; show that the sum of the squares of the distances of the centroids of the triangles BOC, COA, AOB, BO_1C, CO_2A, AO_3B from the centroid of the triangle ABC, is

$$\tfrac{2}{9}(ab+bc+ca)+\tfrac{4}{9}\mathrm{D}(\mathrm{D}-r).$$

Solution by Professor WOLSTENHOLME, M.A.

The distance between the centres of inertia of the triangles ABC, BOC, $=\frac{1}{3}$OA; and similarly for each of the others; hence, sum of squares

$$= \tfrac{1}{9}(\mathrm{AO}^2+\mathrm{AO}_1^2+\dots) = \tfrac{1}{9}[(4\mathrm{R}\sin\tfrac{1}{2}\mathrm{B}\sin\tfrac{1}{2}\mathrm{C})^2+(4\mathrm{R}\cos\tfrac{1}{2}\mathrm{B}\cos\tfrac{1}{2}\mathrm{C})^2+\dots]$$

$$= \tfrac{1}{9}[8\mathrm{R}^2(1+\cos\mathrm{B}\cos\mathrm{C})+\dots+\dots]$$

$$= \tfrac{8}{9}\mathrm{R}^2(\sin\mathrm{B}\sin\mathrm{C}+\dots)+\tfrac{8}{3}\mathrm{R}^2+\tfrac{8}{9}\mathrm{R}^2[\cos(\mathrm{B}-\mathrm{C})+\dots]$$

$$= \tfrac{2}{9}(bc+ca+ab)+\tfrac{8}{9}\mathrm{R}^2(3-\cos\mathrm{A}-\cos\mathrm{B}-\cos\mathrm{C})$$

$$= \tfrac{2}{9}(bc+ca+ab)+\tfrac{8}{9}\mathrm{R}^2\left(2-\frac{r}{\mathrm{R}}\right) = \tfrac{2}{9}(bc+ca-ab)+\tfrac{4}{9}(\mathrm{D}^2-\mathrm{D}r).$$

5745. (By S. TEBAY, B.A.)—A heavy egg-shaped shell, whose interior surface is perfectly smooth, is filled with incompressible fluid and placed upon a smooth horizontal plane. If it be twirled about a vertical axis through their common centre of gravity, determine the motion.

Solution by the PROPOSER.

Using the customary notation, Mr. WOOLHOUSE (*Gentleman's Diary* for 1838) has found, for the motion of any solid of revolution,

$$\frac{d\phi}{dt}-\cos\theta\frac{d\omega}{dt}=c,\quad c\cos\theta-\frac{A}{C}\sin^2\theta\frac{d\omega}{dt}=c',$$

$$A\left\{\sin^2\theta\left(\frac{d\omega}{dt}\right)^2+\left(\frac{d\theta}{dt}\right)^2\right\}+M\left\{\left(\frac{dz}{dt}\right)^2+2gz\right\}=c'';$$

c, c', c'' being constants depending on initial motion, M the mass of the solid, z the perpendicular from the centre of gravity on the plane, and ω the inclination to x of the line from the foot of z to the point of contact. For M write $M+m$, m being the mass of the fluid; then

$$A\left\{\sin^2\theta\left(\frac{d\omega}{dt}\right)^2+\left(\frac{d\theta}{dt}\right)^2\right\}+(M+m)\left\{\left(\frac{dz}{dt}\right)^2+2gz\right\}=c''.$$

If m be small in comparison with M, the motion corresponds to that of a hard-boiled egg. If m be large compared with M, we have

$$\left(\frac{dz}{dt}\right)^2+2gz=c'';$$

and as no motion is communicated to m, $\frac{dz}{dt}=0$, and z is constant.

[This question took its rise from Col. CLARKE's ingenious problem (5700), to find, *by a mathematical process*, whether a boiled egg is hard or soft, two solutions of which have been given in *Reprint*, Vol. XXX., pp. 30, 42. Mr. TEBAY's investigation, as given above, has been objected to "as no solution at all," inasmuch as, besides other defects, "there is no quality of fluids taken into consideration."]

5942. (By C. LEUDESDORF, M.A.)—Show that the equation of the segment of a circle which passes through (x_1, y_1), (x_2, y_2), and contains an angle θ, is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=\pm\cot\theta\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1\end{vmatrix}$.

Solution by G. TURRIFF, M.A.; C. K. PILLAI, M.A.; *and others.*

Let (α, β) be a point on the segment; then the equations of the lines joining (α, β) with (x_1, y_1), (x_2, y_2) are

$$y-\beta=\frac{y_1-\beta}{x_1-\alpha}(x-\alpha),\quad\text{and}\quad y-\beta=\frac{y_2-\beta}{x_2-\alpha}(x-\alpha);$$

therefore $\pm\tan\theta=\left\{\frac{y_1-\beta}{x_1-\alpha}-\frac{y_2-\beta}{x_2-\alpha}\right\}\div\left\{1+\frac{(y_1-\beta)(y_2-\beta)}{(x_1-\alpha)(x_2-\alpha)}\right\};$

hence we have $(x_1-\alpha)(x_2-\alpha)+(y_1-\beta)(y_2-\beta)$

$$=\pm\cot\theta\left\{(y_1-\beta)(x_2-\alpha)-(y_2-\beta)(x_1-\alpha)\right\}=\&c.$$

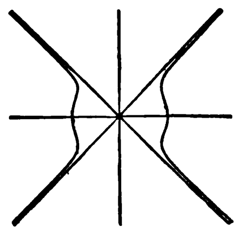
6142. (By W. H. H. Hudson, M.A.)—**Trace the curve**

$$r\,(\cos 2\theta)^{\frac{1}{2}} = 2a\cos^2\theta,$$

and find its points of inflexion.

Solution by G. Heppel, M.A.; C. Bickerdike, B.A.; *and others.*

It is clear that the curve is included between the lines $\theta = \frac{1}{4}\pi$ and $\theta = -\frac{1}{4}\pi$, that these lines are asymptotes, and that there is a similar branch on the negative side of the origin. To find a point of inflexion, put $v = \cos 2\theta$; then we have



$$\frac{dv}{d\theta} = -2\,(1-v^2)^{\frac{1}{2}},\quad \frac{d^2v}{d\theta^2} = -4v,$$

$$\frac{a}{r} = au = \frac{v^{\frac{1}{2}}}{1+v},\quad a\frac{du}{dv} = \frac{1-v}{2\,(1+v)^2\,v^{\frac{1}{2}}},$$

$$a\frac{d^2u}{dv^2} = \frac{3v^2-6v-1}{4\,(1+v)^3\,v^{\frac{3}{2}}},$$

$$\frac{d^2u}{d\theta^2} = \frac{d^2u}{dv^2}\cdot\left(\frac{dv}{d\theta}\right)^2 + \frac{du}{dv}\cdot\frac{d^2v}{d\theta^2};$$

and at a point of inflexion $u + \dfrac{d^2u}{d\theta^2} = 0$; therefore $8v^2-5v-1 = 0$,

whence $\qquad v = \cos 2\theta = \frac{1}{16}(5+\sqrt{57})$, and $\theta = 19^\circ\,10'$.

From 0° to 20° the curve is almost coincident with (being just outside) a circle whose radius is $2a$.

[Since $r\,(\cos 2\theta)^{\frac{1}{2}} = 2a\cos\theta = a\,(1+\cos 2\theta)$, or $r = a\,(\sec 2\theta)^{\frac{1}{2}} + a\,(\cos 2\theta)^{\frac{1}{2}}$, each radius of the curve is an arithmetic mean between the corresponding radii of the rectangular hyperbola $r = 2a\,(\sec 2\theta)^{\frac{1}{2}}$, and the lemniscate $r = 2a\,(\cos 2\theta)^{\frac{1}{2}}$; hence, from the figure, it is clear that there must be points of inflexion, and they may be accordingly found in the usual way.]

6136. (By the Rev. W. A. Whitworth, M.A.)—Each of two urns contains n cards numbered successively 1, 2, 3, ... n. I draw a card from each urn and receive a number of shillings equal to the difference between the numbers on the two cards drawn. I repeat this operation, again receiving the difference in shillings between the two numbers which I draw; and so on, until at the n^{th} operation the cards are exhausted. Show that my expectation is worth $\frac{1}{3}(n^2-1)$ shillings.

Solution by the Rev. J. L. Kitchin, M.A.; G. Heppel, M.A.; *and others.*

The number of cases where the card r in the first urn comes out in combination with the card s in the second, is $(n-1)\,!$, and the corresponding money value is $(n-1)\,!\,(r \sim s)$. Hence the money value of the total of cases, where r stands first, is

$$(n-1)\,!\,\{0+1+2+3+\ldots+(r-1)\,!+1+2+3+\ldots+(n-r)\,!\}$$
$$= (n-1)\,!\,\{\tfrac{1}{2}r\,(r-1)\,!+\tfrac{1}{2}(n-r+1)(n-r)\}$$
$$= \tfrac{1}{2}(n-1)\,!\,\{n^2+n-2\,(n+1)\,r+2r^2\}.$$

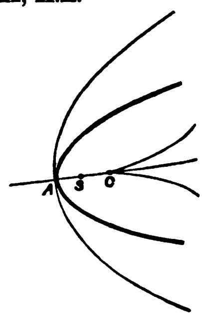
Hence, giving r all values from 1 to n, the total of all money values is

$$\tfrac{1}{4}(n-1)!\left\{n^3+n^2-n(n+1)^2+\tfrac{1}{3}n(n+1)(2n+1)\right\} = \tfrac{1}{6}(n+1)!(n-1);$$

and of these values I receive some one out of $n!$ sets; therefore my expectation is worth $\frac{1}{6}(n^2-1)$ shillings.

6043. (By the EDITOR.)—Find the locus of the centres of circles tangential to a parabola and the tangent at its vertex.

I. *Solution by* Professor WOLSTENHOLME, M.A.



The equation of the locus is

$$y^4-y^2(2x^2+18ax+27a^2)+x^3(x+2a) = 0,$$

the equation of the parabola being

$$y^2 = 4a(x+2a).$$

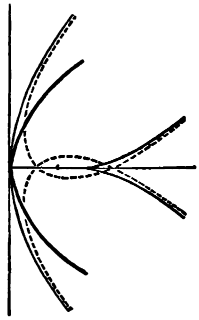
There is a cusp at the origin, which is the centre of curvature of the vertex of the parabola, and there are the two parabolic asymptotes

$$(\pm y+x+\tfrac{3}{2}a)^2 = 2a(\mp y+x+\tfrac{3}{2}a).$$

The form of the curve is indicated by the fainter lines; where A, S are the vertex and focus of the given parabola, and C is the centre of curvature at the vertex.

II. *Solution by* G. HEPPEL, M.A.; CHRISTINE LADD; *and others.*

In the figure the thick lines represent the parabola and its tangent, the thin lines are the required locus, and the dotted lines are parabolic asymptotes.



It is clear from the conditions of the question that the axis of x is a part of the locus; and, to find the rest, let $y^2 = 4ax$ be the parabola; (x, y) the centre of the circle; and (h, k) the point where it touches the parabola; then, to eliminate $h, k, \frac{dk}{dh}$, we have the following equations:—

$$(x-h)^2+(y-k)^2 = x^2, \qquad k^2 = 4ah,$$

$$x-h+(y-k)\frac{dk}{dh} = 0, \qquad k\frac{dk}{dh} = 2a.$$

The result is $2y^2 = 2x^2+10ax-a^2 \pm a^{\frac{1}{2}}(a+4x)^{\frac{3}{2}}$,

or $\quad y^4-y^2(2x^2+10ax-a^2)-x(2a-x)^3 = 0.$

From the second form we see that, when x is less than $2a$, there is one

positive value of y^2; and when x is greater than $2a$, there are two. Also, if $x = 2a$, $y = 0$, and $\frac{dy}{dx} = \frac{0}{0}$; hence there is a cusp at this point.

From the first form, expanding y^2 in descending powers of x, we have

$$y^2 = x^2 \pm 4a^{\frac12} x^{\frac32} + 5ax \pm \tfrac32 a^{\frac32} x^{\frac12} - \tfrac12 a^2 \pm \tfrac38 a^{\frac52} x^{-\frac12} + \&c.;$$

therefore $\pm y = x \pm 2a^{\frac12}x^{\frac12} + \tfrac12 a \mp \tfrac14 a^{\frac32} x^{-\frac12} + \tfrac18 a^2 x^{-1} + \&c.,$

therefore $\pm y - x - \tfrac12 a = \pm 2a^{\frac12}x^{\frac12} \mp \tfrac14 a^{\frac32} x^{-\frac12} + \tfrac18 a^2 x^{-1} + \&c.,$

therefore $(\pm y - x - \tfrac12 a)^2 = 4ax - a^2$ are parabolic asymptotes.

These have their vertices at a point $x = \tfrac12 a$, $y = 0$, and have their axes inclined at an angle of 45° to the axis of x. The latus rectum of each is $a\sqrt{2}$. They cut the axis of x again at a point $x = \tfrac52 a$.

6184. (By G. Heppel, M.A.) — Find the number of terms in the expansion of the k^{th} power of the sum of n different quantities.

Solution by the Rev. J. L. Kitchin, M.A.; Christine Ladd; *and others.*

The number of terms will be the number of combinations of the n things $a, b, c, \ldots$ (including repetitions) taken k together; and this is $N_{n,k} = \frac{(n+k-1)!}{(n-1)!\,k!}$, a well-known result.

6048 & 6086. (By J. Hammond, M.A.)—Prove that the n^{th} convergent to $2 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+\ldots}$ is $\frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n - (1-\sqrt{2})^n}$;

and to $2 - \frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\,\ldots$ is $\frac{n+1}{n}$.

Solution by Prof. Tanner, M.A.; R. Knowles, B.A., L.C.P.; &c.

(Quest. 6048.) If u_n represent the n^{th} convergent, we have $u_n = 2 + \frac{1}{u_{n-1}}$.

Put $\frac{v_{n+1}}{v_n} = u_n$; then the equation becomes $v_{n+1} = 2v_n + v_{n-1}$; giving

$$v_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n,$$

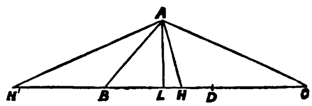
and $$u_n = \frac{c_1(1+\sqrt{2})^{n+1} + c_2(1-\sqrt{2})^{n+1}}{c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n}.$$

But $u_1 = 2$, so that $c_1 + c_2 = 0$; and this gives the required form.

(Quest. 6086.) The converging fractions in this case are $\frac{2}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, &c.; and the law of formation gives us $a_n = 1 + b_n$; and since $b_1 = 1$, $b_2 = 2, \ldots$ $b = n_n$, therefore $a_n = 1 + n$, and $\dfrac{a_n}{b_n} = \dfrac{n+1}{n}$.

6178. (By E. Anthony, M.A.)—Let D be the middle of the base BC of a triangle ABC; AL the perpendicular from A on BC; and AH, AH′ the bisectors of the angle A; then prove that DL . HH′ = AB . AC.

Solution by A. L. Selby, M.A.; E. Rutter; *and others.*



$$\begin{aligned} AB \,.\, AC &= BH \,.\, HC + AH^2 \\ &= BH \,.\, HC + HL \,.\, HH'; \end{aligned}$$

and since P $\{BHCH'\}$ is harmonic,

we have $DH \,.\, DH' = DC^2$;

therefore $DH \,.\, HH' = BH \,.\, HC$,

and $$\begin{aligned} AB \,.\, AC &= DH \,.\, HH' + HL \,.\, HH' \\ &= DL \,.\, HH'. \end{aligned}$$

6116. (By T. Morley, L.C.P.)—Solve the equations

$$x^3 - a = y^3 - b = z^3 - c = xyz.$$

Solution by E. W. Symons, B.A.; T. R. Terry, M.A.; *and others.*

Eliminating y and z, we have

$$(a+b+c)\,x^6 - (2a^2 + ab + ac - bc)\,x^3 + a^3 = 0;$$

$$\therefore\quad 2\,(a+b+c)\,x^3 = 2a^2 + ab + ac - bc \mp \left[a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a+b+c)^{\frac{1}{2}}\right].$$

The values of y and z follow from symmetry.

5272. (By Professor Townsend, F.R.S.)—A solid of revolution of uniform density, turning without friction round a fixed point on its axis of figure, and rolling without sliding on a fixed plane inclined at any angle to the horizon, being supposed to have the velocity of rotation just sufficient to carry it with exhausted energy of motion to its position of

unstable equilibrium against the action of gravity; determine, in finite terms, its time of passage from the opposite position of stable equilibrium to any other on its way.

Solution by the PROPOSER.

Denoting by O the fixed point, by OP and OV the perpendicular and the vertical from it to the plane, by C the centre of inertia of the body, by X its point of contact with the plane in any position, by A and B the positions of X on the line PV which correspond to those of stable and unstable equilibrium of the body, by θ and ϕ the two variable angles APX and COV, by ω the variable velocity of rotation round the instantaneous axis OX, by k the constant arm of inertia round OX, by α the fixed angle POV of inclination of the plane, by β and γ the two constant angles COP and COX, by l the constant distance OC, and by m the mass of the body; then, since, for any two positions of the body, by the equation of *vis viva*, $k^2(\omega^2-\omega_1^2) = 2gl(\cos\phi-\cos\phi_1)$, since, by the geometry of rotation, $\omega\sin\gamma = \frac{d\theta}{dt}\sin\beta$, and since, by spherical trigonometry, $\cos\phi = \cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\theta$; therefore, for any two positions of the body, $\frac{d\theta^2}{dt^2}-\frac{d\theta_1^2}{dt^2} = 2\frac{g}{l'}.(\cos\theta-\cos\theta_1)$,

where $$l' = \frac{k^2}{l}\cdot\frac{\sin\beta}{\sin\alpha}\cdot\operatorname{cosec}^2\gamma.$$

Hence, when, as by hypothesis, $\frac{d\theta_1}{dt} = 0$ when $\theta_1 = \pi$, then

$$\frac{d\theta^2}{dt^2} = 2\frac{g}{l'}(1+\cos\theta) = 4\frac{g}{l'}\cos^2\tfrac{1}{2}\theta;$$

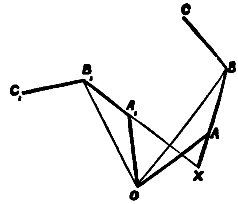
and therefore, by integration in finite terms, supposing θ and t to commence together, $t = \left(\frac{l'}{g}\right)^{\frac{1}{2}}\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)$;

which accordingly is the required value, in finite terms, for t in terms of θ.

Putting $\theta=\pi$ in the above formula, we get $t=\infty$; which accords with the general property that, under circumstances similar to the above, a position of unstable equilibrium can be reached with exhausted energy of motion by a body or system of bodies against the action of the forces, only at the expiration of an infinite time.

5264. (By the REV. H. G. DAY, M.A.)—Two similar polygons ABCDE..., $A_1B_1C_1D_1E_1$... are similarly situated on lines AB, A_1B_1 (not parallel); show that (1) if they are situated towards the same parts, a point O can be found at which AA_2, BB_1, CC_1, &c. all subtend the same angle; and (2) if towards opposite parts, a point O can be found at which the bisectors of the angles AOA_1, BOB_1, COC_1, coincide.

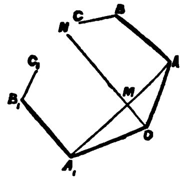
Solution by PROPOSER.



1. Produce BA, B_1A_1 to meet in X; and draw circles round AA_1X, BB_1X to intersect in O. Then, the triangles BOA, B_1OA_1, being equiangular, are similar; and O is similarly situated in respect to the two figures ABCDE, $A_1B_1C_1D_1E_1$;

hence, $\angle BOC = B_1O_1C_1$;

therefore $COC_1 = BOB_1 = AOA_1$, &c.



2. Divide AA_1 in M, so that

$$AM : A_1M = AB : A_1B;$$

and through M draw MN, equally inclined to AB and A_1B_1. Then MN is evidently symmetrically placed as regards the two polygons. Draw AO, A_1O to meet in MN at equal angles with it; then the angle $OAB = OA_1B_1$; and O is symmetrically situated with regard to the two polygons, therefore ON bisects BOB_1, COC_1, &c.

[In (1) the angle which any two homologous points subtend is the angle between any two homologous sides of the polygon; that is to say, it is the angle through which either polygon must be turned to bring the two into corresponding positions. From the above it is clear that two similar curves can always be expressed by one of the systems of equations

$$r = af(\theta),\quad \rho = bf(\theta-\alpha),\quad \text{or}\quad r = af(\theta),\quad \rho = bf(\alpha-\theta)].$$

6111. (By A. JOHNSON, M.A.)—A raft, whose length, breadth, and depth are l, b, d feet respectively, is made of wood, whose specific gravity is s; find the utmost weight it will bear before sinking.

Solution by E. W. SYMONS, B.A.; E. RUTTER; *and others.*

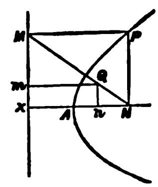
Supposing the raft to be just totally immersed when a weight W is placed on it (without altering shape and volume of raft), we have, equating weight of raft to fluid pressure,

$$glbd \,.\, s + W = glbd, \quad \text{therefore} \quad W = glbd\,(1-s).$$

6180. (By R. E. RILEY, B.A.)—From any point P of a parabola PM, PN are drawn perpendicular to the directrix and axis. If Q be a point in MN, dividing it in any constant ratio, prove that the locus of Q is a co-axal parabola.

Solution by R. Knowles, B.A., L.C.P.; D. Edwardes; *and others.*

Let $MQ = m \,.\, QN$, and let Q be (X, Y); then we have
$$\frac{m}{1+m} = \frac{X}{x}, \quad 1+m = \frac{y}{Y},$$
the origin being the foot (X) of the directrix; but
$$y^2 = 4a(x-a),$$
therefore $(1+m)^2 Y^2 = 4a\left\{(1+m^{-1})X - a\right\}$, which proves the theorem.



2756. (By J. Griffiths, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribed nine-point, and self-conjugate circles as a given triangle.

Solution by Professor Wolstenholme, M.A.

Let ABC be the given triangle, P any point on the circumscribed circle, and let the polar of P, with respect to the self-conjugate circle, meet the circumscribed in QR; then PQR will also be self-conjugate; for, if not, let R′ be the pole of PQ, then R′ must lie on QR, the polar of P, and the triangle PQR′ is self-conjugate to the circle. But ABC is also; therefore the six points A, B, C, P, Q, R′ lie on one conic, and five of them are on the circumscribed circle; therefore R′ is also on that circle, and coincides with R.

If the circumscribed circle and self-conjugate circle of a triangle be given, so also is the nine-point circle, since its centre bisects the distance between the two given centres, and its radius is half one of the given radii.

The sides will obviously all touch the hyperbola, of which the two given centres are foci, whose major axis is equal to the radius of the given circumscribed circle, and the nine-points circle will be its auxiliary circle.

[A solution by Prof. Cayley is given on p. 98 of Vol. X. of the *Reprint*.]

6045. (By R. F. Davis, M.A.)—APQ is a triangle similar to a given fixed triangle. If A be fixed, and the locus of P a straight line, show that the locus of Q is also a straight line.

Solution by T. Morley, L.C.P.; H. Murphy; *and others.*

This theorem is obvious. We have two radii AP, AQ whose ratio is given, and the angle between them. If, then, P describe any locus, Q will

describe a similar locus, turned through an angle PAQ. If (P) be a conic given by the general equation, and $\angle$PAQ $= a$, and AP $= m$. AQ, the conic (Q) is easily seen to be

$$m^2x^2 (a \cos^2 a + b \sin^2 a + h \sin 2a)\ 2y$$
$$+ m^2y^2 (a \sin^2 a + b \cos^2 a - h \sin 2a) + 2m^2xy \left[\tfrac{1}{2}(b-a) \sin 2a + h \cos 2a\right]$$
$$+ 2mx (g \cos a + f \sin a) + 2my (f \cos a - g \sin a) + c = 0.$$

[By reference to the Editorial Note on the Solution of Quest. 4149, on p. 49 of Vol. XXII. of the *Reprint*, the same general result will be seen to follow at once. Mr. DAVIS states that the question was suggested by 683 in the 2nd edition of Prof. WOLSTENHOLME's collection of Problems.]

5355. (By S. M. DRACH, F.R.A.S.)—
Hindoo chess-inventors' claim bred despair;
'Twas doubling wheat-grains on following square.
How long 'twould feed mankind I here declare.
Four hundred millions' daily quartern loaf,
Ta'en from this fund, I find would be enough
From Noah's deluge human folk to sate,
Past one-nine-eighteen of the Christian date.
If useless Bacco-Bacchus men don't inhale,
Good corn-bread would suffice up to this tale.

Solution by the PROPOSER.

Grains $2^{64}-1$; in a quarter $512 \times 256 \times 30$; quartern loaves $577 \times 4\frac{11}{32}$, Mackenzie's Tab., p. 16; result, $(2^{52} \times 577)$ divided by $(1461 \times 417 \times 10^8)$, or $4265\frac{1}{4}$ years, or $2347 + 1918$.

5998. (By C. LEUDESDORF, M.A.)—A lamina in the form of an ellipse has one point P in its circumference fixed, and is struck perpendicularly to its plane at another point Q in the circumference; show that, if it begin to rotate about the tangent to the ellipse at P, then PQ is a diameter.

Solution by T. R. TERRY, M.A.; Prof. MATZ, M.A.; *and others.*

Taking the lines through P parallel to the axes as axes of x and y, let ω_x, ω_y be the angular velocities round these axes generated by the blow B; then, taking moments round these axes, and putting M the mass of the lamina, we have

$$M\left(\tfrac{1}{4}b^2 + b^2 \sin^2\theta\right)\omega_x - Mab \sin\theta \cos\theta\, \omega_y = Bb(\sin\phi - \sin\theta) \;......(1),$$
$$M\left(\tfrac{1}{4}a^2 + a^2 \cos^2\theta\right)\omega_y - Mab \sin\theta \cos\theta\, \omega_x = -Ba(\cos\phi - \cos\theta) \;...(2).$$

If the body begins to turn round the tangent at P, we have

$$\frac{\omega_x \cos\theta}{a}+\frac{\omega_y \sin\theta}{b}=0, \quad\text{or}\quad \frac{b\omega_x}{a\omega_y}=\frac{\sin\theta}{-\cos\theta};$$

therefore, from (1) and (2), by division, we obtain

$$\frac{(\frac{1}{2}+\sin^2\theta)\sin\theta+\sin\theta\cos^2\theta}{-(\frac{1}{2}+\cos^2\theta)\cos\theta-\sin^2\theta\cos\theta}=\frac{\cos\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+\phi)};$$

therefore $\sin\theta\sin\frac{1}{2}(\theta+\phi)+\cos\theta\cos\frac{1}{2}(\theta+\phi)=0$, or $\theta-\phi=\pi$;
hence PQ is a diameter of the ellipse.

5815. (By S. TEBAY, B.A.)—Property worth £400 is mortgaged for s pounds, and afterwards sold out for an integral number (t) of pounds, which is s per cent. on the mortgage. If $s+t$ be > £229, but < £335, find the probable amount realized by the transaction.

Solution by the PROPOSER.

First, $\dfrac{t}{s}=\dfrac{s}{100}$; or $s=10\,(t)^{\frac{1}{2}}$. Let $u_t=10\,(t)^{\frac{1}{2}}+t$; then

$$\Sigma u_{t+1}=\int u_{t+1}\,dt-\tfrac{1}{2}u_{t+1}+\frac{B_1}{2\,!}\,\frac{du_{t+1}}{dt}-\frac{B_3}{4\,!}\,\frac{d^3u_{t+1}}{dt^3}+\&c.+C$$

(B_1, B_3, &c. being Bernoulli's numbers)

$$=\tfrac{20}{3}(t+1)^{\frac{3}{2}}+\tfrac{1}{2}(t+1)^2-\tfrac{1}{2}\left[10\,(t+1)^{\frac{1}{2}}+t+1\right]+\tfrac{1}{12}\left[5\,(t+1)^{-\frac{1}{2}}+1\right]$$
$$-\tfrac{1}{192}(t+1)^{-\frac{5}{2}}+\&c.+C.$$

The limits of t are 120 and 195; therefore, the average required is

$$\tfrac{1}{75}\,\Sigma_{t=120}^{t=195}u_{t+1}=\tfrac{1}{75}\left[\tfrac{20}{3}(14^3-11^3)+\tfrac{1}{2}(196^2-121^2)\right.$$
$$\left.-\tfrac{1}{2}(140+196-110-121)+\tfrac{1}{2}\left(\tfrac{5}{14}+1-\tfrac{5}{11}-1\right)\right]=£283.\ 5s.\ 4d.$$

5977. (By EDWYN ANTHONY, M.A.)—If θ, θ' be the eccentric angles corresponding to the extremities of the chord PQ of an ellipse, which is normal at P, $\qquad \tan\theta\tan\frac{1}{2}(\theta+\theta')=e^2-1.$

Solution by R. GRAHAM, M.A.; A. W. SCOTT, M.A.; *and others.*

The equations of PQ and the tangent at P are

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\theta')+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\theta')=\cos\tfrac{1}{2}(\theta-\theta'),\quad \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1;$$

and, since PQ is a normal, these lines are at right angles;

therefore $$\frac{\cos\theta\cos\frac{1}{2}(\theta+\theta')}{a^2}+\frac{\sin\theta\sin\frac{1}{2}(\theta+\theta')}{b^2}=0,$$

or $$\tan\theta\tan\frac{1}{2}(\theta+\theta')=-\frac{b^2}{a^2}=e^2-1.$$

6168. (By the EDITOR.)—A coin is dropped at random over an indefinite horizontal grating formed of parallel equidistant wires; prove that, according as the distance between each pair of wires is equal to (1) the circumference, (2) the diameter, (3) the radius, of the coin, the respective chances that the coin will strike in passing through the grating are

$$p_1=\frac{1}{4},\quad p_2=\frac{1}{4}\pi,\quad p_3=\frac{1}{6}\pi+\frac{1}{4}\sqrt{3}.$$

I. *Solution by* C. J. MONRO, M.A.

Neglect the thickness of the coin and wire, and, in comparison with its velocity of translation, the coin's velocity of rotation. Let a represent the distance between adjacent wires in magnitude or direction, and let h be the coin's breadth parallel to a; then the chance, for a particular value of h, is $\frac{h}{a}$, when $h<a$, and 1 when $h>a$.

Let c be the diameter of the coin, and α the inclination of its normals to a, then $h=c\sin\alpha$. Therefore, if $c\sin A=a$, and $Q\,d\alpha$ be the chance that the inclination is between α and $\alpha+d\alpha$, the chance required (call it p)

is $\frac{c}{a}\int_0^{\frac{1}{2}\pi}Q\sin\alpha\,d\alpha$, if $a>c$; and $\frac{c}{a}\int_0^{A}Q\sin\alpha\,d\alpha+\int_A^{\frac{1}{2}\pi}Q\,d\alpha$, if $a<c$.

As the coin is dropped at random, its plane is disposed at random, and its normals directed at random. When, in such cases, all depends on inclination to a plane fixed in disposition (as here, a plane perpendicular to a), there is a tendency to distribute the inclinations uniformly. See COURNOT'S *Exposition de la Théorie des Chances*, p. 270, on the inclinations of the planes of the cometary orbits. But random distribution requires geometrical invariance, or independence of all fixed loci; and it seems to follow that the solid angle of a cone, divided by its integral, is the measure of the chance that a direction taken at random shall fall within the range bounded by such cone. Then $Q=\sin\alpha$, and p is

$$\frac{\pi}{4}\cdot\frac{c}{a},\ \text{ or }\ \frac{\pi}{4}\sin A,\quad \text{if } a>c;$$

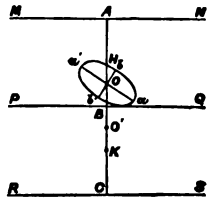
and $$\frac{1}{2a}\left\{c\sin^{-1}\frac{a}{c}+(c^2-a^2)^{\frac{1}{2}}\right\},\ \text{ or }\ \frac{1}{2}\left(\frac{A}{\sin A}+\cos A\right),\quad \text{if } a<c.$$

These formulas will give the Proposer's results.

II. *Solution by* Professor SEITZ, M.A.

Let MN, PQ, RS represent three consecutive wires of the grating; ABC a perpendicular intersecting them in A, B, C; H and K the middle points of AB and BC.

It is evidently necessary to consider only those cases in which the centre of the coin in a free fall would pass through some point of HK. Let the ellipse $aba'b'$ be the projection of the circumference of the coin on the plane of the grating when it just touches the wire PQ in passing through the grating, aa' and bb' being the axes of the ellipse, and the centre O being in HK. Take BO′ equal to BO. Let θ = the angle with which the coin meets the grating, r = the radius of the coin, $OO' = x$, and $\angle BOb' = \phi$.



Then we have $Oa = r$, $Ob = r\cos\theta$, and since OB is the perpendicular from the centre on the tangent PQ, $x = 2r\,(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}$.

1. Let HK $= 2\pi r$; then we have

$$p_1 = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} x\,d\phi\sin\theta\,d\theta \div \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} 2\pi r\,d\phi\sin\theta\,d\theta$$

$$= \frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\,d\phi\sin\theta\,d\theta$$

$$= \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\left\{1+\cos\phi\log\tan\tfrac{1}{2}\phi-\sec\phi\log\tan\tfrac{1}{2}\phi\right\}d\phi$$

$$= \frac{1}{\pi^2}\left\{\phi+\sin\phi\log\tan\tfrac{1}{2}\phi-\phi\right\}_0^{\frac{1}{2}\pi}-\frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\sec\phi\log\tan\tfrac{1}{2}\phi\,d\phi$$

$$= \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}(1+\tfrac{1}{3}\cos^2\phi+\tfrac{1}{5}\cos^4\phi+\tfrac{1}{7}\cos^6\phi+\dots)\,d\phi$$

$$= \frac{1}{\pi^2}\cdot\frac{\pi}{2}\left(1+\tfrac{1}{2}\cdot\tfrac{1}{3}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{7}+\dots\right)=\frac{1}{\pi^2}\cdot\frac{\pi^2}{4}=\frac{1}{4}.$$

2. Let HK $= 2r$; then we have $p_2 = \pi p_1 = \frac{1}{4}\pi$.

3. Let HK $= r$; then, if $\phi < \frac{1}{6}\pi$ and $\theta > \sin^{-1}(\frac{1}{2}\sqrt{3}\,.\,\sec\phi)$, or θ_1, the number of ways the coin can fall so as to strike in passing, is x; but if $\phi < \frac{1}{6}\pi$ and $\theta < \theta_1$, or $\phi > \frac{1}{6}\pi$ and θ has any value, the number of ways is r. Hence we have

$$p_3 = \frac{\int_0^{\frac{1}{6}\pi}\left(\int_0^{\theta_1} r\sin\theta\,d\theta+\int_{\theta_1}^{\frac{1}{2}\pi}x\sin\theta\,d\theta\right)d\phi+\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}r\,d\phi\sin\theta\,d\theta}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}r\,d\phi\sin\theta\,d\theta}$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{6}\pi}\left\{\int_0^{\theta_1}\sin\theta\,d\theta+\int_{\theta_1}^{\frac{1}{2}\pi}2(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\sin\theta\,d\theta\right\}d\phi+\frac{2}{\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\phi\sin\theta\,d\theta$$

$$= \frac{2}{3}+\frac{1}{\pi}\int_0^{\frac{1}{6}\pi}\left\{2-(1-\tfrac{3}{4}\sec^2\phi)^{\frac{1}{2}}+2(\cos\phi-\sec\phi)\log\left[\frac{2\sin\phi}{1+(1-4\sin^2\phi)^{\frac{1}{2}}}\right]\right\}d\phi$$

$$= \frac{\sqrt{3}}{4}-\frac{2}{\pi}\int_0^{\frac{1}{6}\pi}\sec\phi\,d\phi\log\left[\frac{2\sin\phi}{1+(1-4\sin^2\phi)^{\frac{1}{2}}}\right].$$

Let $2\sin\phi = \sin\psi$; then we have

$$p_3 = \tfrac{1}{4}\sqrt{3} - \frac{8}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\cos\psi}{4-\sin^2\psi} \log\tan\tfrac{1}{2}\psi\, d\psi$$

$$= \tfrac{1}{4}\sqrt{3} + \frac{2}{\pi}\left\{\log\left(\frac{2-\sin\psi}{2+\sin\psi}\right)\log\tan\tfrac{1}{2}\psi\right\}_0^{\frac{1}{2}\pi} - \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \operatorname{cosec}\psi\, d\psi \log\left(\frac{2-\sin\psi}{2+\sin\psi}\right)$$

$$= \tfrac{1}{4}\sqrt{3} + \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2^3}\sin^2\psi + \frac{1}{5}\cdot\frac{1}{2^5}\sin^4\psi + \frac{1}{7}\cdot\frac{1}{2^7}\sin^6\psi + \dots\right)d\psi$$

$$= \tfrac{1}{4}\sqrt{3} + \left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{1}{2^5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}\cdot\frac{1}{2^7} + \dots\right)$$

$$= \tfrac{1}{8}\pi + \tfrac{1}{4}\sqrt{3}.$$

[A *general* solution by the PROPOSER is given on pp. 56–58 of Vol. I. of the *Reprint*.]

1433. (By the EDITOR.)—Prove the following reciprocal cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

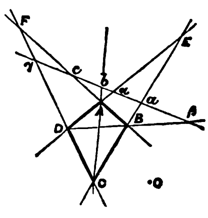
a′. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b′. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution.

Solution by the late Professor CLIFFORD, F.R.S.

Six points are in involution when the anharmonic ratio of any four is equal to that of their four conjugates.



a. Let then ABC be the triangle, D the point of concourse; and let a straight line meet the sides and corresponding lines in $a, b, c, \alpha, \beta, \gamma$ respectively; then we have

$$[abc\gamma] = \{A\,.\,DCB\gamma\} = \{A\,.\,DCF\gamma\}$$
$$= \{B\,.\,DCF\gamma\} = [\beta ac\gamma] = [a\beta\gamma c],$$

which proves the proposition.

a′. Let now ADF be the triangle, C, B, E the collinear points, and take any point O; then we have

$$\{O\,.\,ABDE\} = \{B\,.\,AODE\} = \{B\,.\,FODC\}$$
$$= \{O\,.\,FBDC\} = \{O\,.\,CDBF\},$$

which proves the second case.

b. The line at infinity is cut in involution, by prop. (*a*); hence, lines

parallel to the given six through any point will form a pencil in involution; turn this pencil through a right angle, and it coincides with the perpendiculars.

b'. Any point at infinity is subtended in involution, by (a'); whence the theorem immediately follows.

5868. (By S. Tebay, M.A.)—Two vessels contain mixtures of wine and water; the first m gallons of wine, and a gallons of water; the second n gallons of wine, and b gallons of water; find the probability that a mixture formed at hazard will contain at least an average of the whole wine.

Solution by the Proposer.

Let the mixture consist of x gallons from the first vessel, and y gallons from the second; these contain $\dfrac{m}{m+a}x$ and $\dfrac{n}{n+b}y$ gallons of wine respectively; hence, putting $\dfrac{m}{m+a} = \mathrm{M}$, and $\dfrac{n}{n+b} = \mathrm{N}$, we must have

$$\mathrm{M}x + \mathrm{N}y > \tfrac{1}{2}(m+n).$$

Let $m > n$; then $\frac{1}{2}(m+n) > n$; therefore the limits of y are 0 and $n+b$. For any value of y, the range of x

$$= m + a - \frac{1}{\mathrm{M}}\left\{\tfrac{1}{2}(m+n) - \mathrm{N}y\right\} = \frac{m-n+2\mathrm{N}y}{2\mathrm{M}};$$

therefore, chance $= \displaystyle\int_0^{n+b}(m-n+2\mathrm{N}y)\,dy \div 2\mathrm{M}\,(m+a)(n+b) = \tfrac{1}{2}.$

This result is pretty evident without calculation; for the chance that the mixture contains not less than $\frac{1}{2}(m+n)$ wine must be equal to the chance that it contains not more than $\frac{1}{2}(m+n)$ wine, that is, $\frac{1}{2}$.

5845. (By W. H. Lowry, M.A.)—Show that the equation whose roots are the cubes of the roots of $f(x) \equiv x^n + \mu x^{n-1} + \ldots = 0$, may be written

$$f(y^{\frac{1}{3}})\left\{(\phi_1 y)^2 - \phi_2(y)\cdot\phi_3(y)\right\} = 0;$$

and determine $\qquad \phi_1(y),\ \phi_2(y),\ \phi_3(y).$

Solution by Prof. Scott, M.A.; *the* Proposer; *and others.*

If ω and ω^2 be the imaginary cube roots by unity, we have

$$f(u)\,.\,f(\omega x)\,.\,f(\omega^2 u) \equiv (x^3 - \alpha^3)(x^3 - \beta^3)\ldots,$$

that is, $\qquad Y_1^3+Y_2^3+Y_3^3-3Y_1Y_2Y_3,$

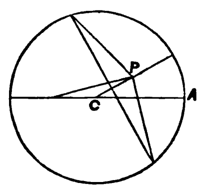
or $(Y_1+Y_2+Y_3)\left\{(Y_1-Y_3)^2-(Y_2-Y_1)(Y_3-Y_2)\right\} \equiv (y-\alpha^3)(y-\beta^3)\ldots\ldots,$

where
$$Y_1 = y^{\frac{1}{3}n}+p_3\,y^{\frac{1}{3}(n-3)}+p_6\,y^{\frac{1}{3}(n-6)}\ldots\ldots,$$
$$Y_2 = p_1y^{\frac{1}{3}(n-1)}+p_7\,y^{\frac{1}{3}(n-4)}+p_7\,y^{\frac{1}{3}(n-7)},$$
$$Y_3 = p_2y^{\frac{1}{3}(n-2)}+p_5\,y^{\frac{1}{3}(n-5)}+p_8\,y^{\frac{1}{3}(n-8)};$$

therefore $\phi_1(y) = Y_1-Y_3$, &c.

5167. (By Prof. SEITZ, M.A.)—A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle; find the chance that the line intersects a given diameter.

Solution by Rev. H. J. DAY, M.A.; E. B. ELLIOTT, M.A.; *and others.*



Let ρ be the distance of the random point P from the centre of the circle C; c the radius of the circle; and ϕ the inclination of CP to the given diameter.

Then the whole range of the angles at which the radius can be drawn as prescribed through p, is $2\cos^{-1}\dfrac{\rho}{2c}$, and the number that cut the diameter in that set is, evidently,

$$\cos^{-1}\frac{\rho}{2c}+\phi-\sin^{-1}\left(\frac{\rho}{c}\sin\phi\right);$$

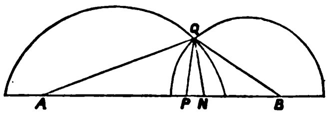
hence the required probability (where $\rho<c$, and $\phi<\frac{1}{2}\pi$)

$$\iint \frac{\phi+\cos^{-1}\left(\dfrac{\rho}{2c}\right)-\sin^{-1}\left(\dfrac{\rho\sin\phi}{c}\right)}{2\cos^{-1}\dfrac{\rho}{2c}}\,\frac{4\rho\,d\rho\,d\phi}{c^2\pi}.$$

[This expression does not seem to admit of integration in finite terms.]

5758. (By Rev. H. G. DAY, M.A.)—If A, B be two fixed points, and P a variable point in the line AB, and if with A, P as foci an ellipse of eccentricity e_1 be drawn, and with B, P as foci an ellipse of eccentricity e_2, intersecting the former in Q; prove that the locus of Q is an ellipse with A, B as foci.

Solution by Professor NASH, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*



Let $AB = 2c$,

$AP = 2\rho$, $PN = x$;

then $AQ = \rho e_1^{-1} + e_1(\rho + x)$,

$PQ = \rho e_1^{-1} - e_1(\rho + x)$,

$BQ = (c-\rho)e_2^{-1} + e_2(c-\rho-x)$,

$PQ = (c-\rho)e_2^{-1} - e_2(c-\rho-x)$;

therefore $x = \dfrac{\rho(1-e_1^2)}{e_1(e_1+e_2)} - \dfrac{(c-\rho)(1-e_2^2)}{e_2(e_1+e_2)}$; hence

$$AQ+BQ = \frac{\rho}{e_1}(1+e_1^2) + \frac{c-\rho}{e_1+e_2}(1+e_2^2) + (e_1-e_2)x$$

$$= \frac{\rho}{e_1+e_2}(2+2e_1e_2) + \frac{c-\rho}{e_1+e_2}(2+2e_1e_2) = \frac{2c(1+e_1e_2)}{e_1+e_2} = \text{a const.}$$

4817. (By Professor SYLVESTER, F.R.S.)—In a school of 15 girls, a rule has been laid down that they shall walk out every day in rows of threes, but that the same two girls shall never come together twice in the same row. The rule is supposed to have been carried out correctly during the six working days of the week, but when the time comes for their going to church together on Sunday it is found to be absolutely impossible to continue it any further. Investigate whether the rule *can* have been correctly carried out during the six previous days; and if so, show how.

Solution by the PROPOSER.

The question *is* possible, and may be answered in various ways.

One solution will leave out the duads 1.2, 1.3, 2.3, 4.5, 4.6, 5.6, 7.8, 7.9, 8.9, 10.11, 11.12, 12.13, 13.14, 14.15, 15.10, which 15 duads evidently cannot be contained in 5 triads, on account of the *cycle*

10, 11, 12, 13, 14, 15.

5403. (By Dr. HOPKINSON, F.R.S.)—An iron hoop, 10 feet in diameter, revolves about an axis through its centre, perpendicular to its plane, with a velocity ω; find the greatest value ω can have without breaking the hoop.

Solution by J. J. WALKER, M.A.

Let T be the tensile strength, w the weight in lbs. weight per cubic inch, R the radius, a the breadth, and b the (small) thickness, of the hoop;

then the critical value of ω is determined by

$$2\pi R\, wab\, \omega^2 R = g\, 2\pi T\, ab, \quad \text{or} \quad \omega^2 = gT \div R^2 w,$$

where g and R are expressed in inches. Taking the tensile strength to be 10 tons weight to the square inch, this would give, for $R = 60$, $\omega = 90$ about; say 14 revolutions per second.

Multiplying the equation above by dR, integrating and reducing, we have the more general result $\omega^2 = gT \div R'R''w$ for a fly-wheel of exterior and interior radii R', R'' respectively.

6215. (By Dr. Francesco Tirelli.)—Siano OA, OB, OC tre corde di un cerchio, sulle quali come diametri si descrivano tre circonferenze C_1, C_2, C_3. Indichiamo con A', B', C' le ulteriori intersezioni di C_2 con C_3, C_3 con C_1, C_1 con C_2. Inoltre C_1 incontri OB ed OC in β e γ, C_2 incontri OC ed OA in γ_1 ed α_1 e C_3 incontri OA ed OB in α_2 e β_2. Dimostrare, che i punti A', B', C' stanno per dritto, e che le rette $\beta\gamma$, $\gamma_1\alpha_1$, $\alpha_2\beta_2$ concorrono in uno stesso punto della $A'B'C'$.

Solution by the Proposer.

Si conducano OA', OB', OC', AC', $C'B$, BA', $A'C$, CB', $B'A$. E facile vedere, che A', B', C' sono i piedi delle perpendicolari condotte da O su' lati del triangolo ABC, perciò stanno per dritto.

Poichè $C'\widehat{A}B' = B\widehat{O}\gamma$, segue $B'C' = \beta\gamma$. E se indichiamo con A'', B'', C'' i piedi delle perpendicolari condotte da' vertici del triangolo ABC su' lati opposti, segue, come si è dedotto $B'C' = \beta\gamma$, che deduconsi $C'A' = \gamma_1\alpha_1$, $A'B' = \alpha_2\beta_2$, ..., cioè che le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, $A'B'C'$ sono tali, che i segmenti compresi fra i tre punti segnati su ciascuna sono uguali. E se X è il punto comune ad $A''\beta\gamma$, $A'B'C'$, si ha

$$XB' \cdot XC' = X\gamma \cdot X\beta;$$

da cui segue pel'osservazione testè fatta $XB' = X\beta$, $XC' = X\gamma$, e quindi, per essere $A''\gamma = A'C'$, si ha $XA'' = XA'$. Adunque le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, $A'B'C'$ sono tali, che i segmenti fra due punti di ciascuna sono uguali, e due di esse si incontrano in un punto che serba uguali distanze da' punti dell' una e dell' altra.

Ciò posto; se ABC, $A'B'C'$, $A_1'B_1'C_1'$ sono tre rette che soddisfanno alle condizioni testè cennate, e che si tagliano due a due in O, O_1, O_2, si ha $OA = OA'$, $O_1A = O_1A_1'$, $O_2A' = O_2A_1'$, da cui agevolmente segue $OO_2 - O_1O_2 = OO_1$ che è assurda. Perciò le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, $A'B'C'$ concorrono in un punto.

[See the Solutions of Quest. 2444, on p. 48 of Vol. VIII. of the *Reprint*.]

5992. (By Elizabeth Blackwood.)—If P, Q, R, S be four random points on the surface of a given sphere; find the chance that the point S will be within the spherical triangle PQR.

Solution by Professor Seitz, M.A.

The required chance is evidently equal to the average area of the triangle PQR, divided by the area of the surface of the sphere. By the solution of Quest. 4849 (*Reprint*, Vol. XXV., p. 64), the average area of the triangle PQR is equal to the area of a tri-rectangular triangle. Hence the required chance is $\frac{1}{8}$.

6098. (By Prof. Cochez.)—Trouver dans le plan de trois cercles un point tel que les tangentes menées de ce point aux troix cercles soient entre elles comme trois droites données.

Solution by J. A. Kealy, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Let $f(x,y) = 0$ and $\phi(x,y)$ be two of the circles; then $f(x,y) = k \cdot \phi(x,y)$ is the locus of a point from which the tangents to the circles are in a given ratio, and this locus must evidently be some circle.

Hence we have the following geometrical construction:—Draw the four common tangents, or at least three of them, to two of the circles; divide them in the given ratio, and draw a circle through the points of division; do the same with another pair; then the intersections of these two circles will give the points required.

5538. (By Professor Townsend, F.R.S.)—A heavy flexible chain, of uniform thickness, being supposed to occupy longitudinally just half the interior of a slender circular tube, of uniform roughness, bounded radially by coaxal horizontal cylinders, and laterally by parallel vertical planes; determine, given all particulars, its two extreme positions of equilibrium under the action of gravity.

Solution by the Proposer.

Denoting by a the radius of the tube, by m the mass per unit of length of the chain, by θ the angle between the radius at any point P of its length and the horizontal radius of the tube through whose extremity it does not pass, by T and R the tension of the chain and the reaction of the tube at P, and by μ and ϵ the coefficient and angle of the friction arising from the uniform roughness of the tube; then, since, by tangential and normal resolution, in the usual manner,

$$\frac{dT}{d\theta} - \mu a R = amg\cos\theta, \text{ and } T - aR = -am\sin\theta;$$

and since, by eliminating R, $\dfrac{dT}{d\theta} - \mu T = amg(\mu\sin\theta + \cos\theta)$;

therefore, by integration, we have in finite terms,

$$T = C\epsilon^{\mu\theta} + \frac{amg}{1+\mu^2}\left[(1-\mu^2)\sin\theta - 2\mu\cos\theta\right],$$

the value of the arbitrary constant C depending, of course, on the particulars of the case.

But, since, in the case in question, $T=0$ when $\theta=\theta_0$ and $=\theta_0+\pi$, where θ_0 = the initial value of θ; therefore, whatever be the value of μ, $C=0$; consequently, for the initial and terminal values of θ,

$$\tan\theta_0 = \tan(\theta_0+\pi) = \frac{2\mu}{1-\mu^2} = \tan 2\epsilon;$$

therefore $\theta_0 = 2\epsilon$; and therefore, &c.

6125. (By Professor MONCK, M.A.)—Find examples of right-angled triangles, with commensurable sides, such that (1) one, (2) neither of the acute angles is less than $\sin^{-1}\frac{3}{5}$.

Solution by G. HEPPEL, M.A.; Rev. W. A. WHITWORTH, M.A.; *and others.*

1. The following are examples of such right-angled triangles:—

15 : 8 : 17, 35 : 12 : 37, 63 : 16 . 65, 99 : 20 : 101, 39 : 80 : 89, 5 : 12 : 13, 7 : 24 : 25, 9 : 40 : 41, 11 : 60 : 61, 45 : 28 : 53, 33 : 56 : 65, 13 : 84 : 85, 77 : 36 : 85.

2. If the hypotenuse be less than 100, the following are the only triangles in which neither of the acute angles is less than $\sin^{-1}\frac{3}{5}$:—

3 : 4 : 5, 21 . 20 : 29, 65 : 72 : 97, 55 : 48 : 73.

6146. (By the Rev. W. ROBERTS, M.A.)—Two tangents to a given parabola make angles with the axis such that the product of the tangents of their halves is constant; prove that the locus of the intersection of the tangents is a confocal parabola.

Solution by E. W. SYMONS, B.A.; R. KNOWLES, L.C.P.; *and others.*

The relation $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta = k$ is equivalent to

$$4k^2(\tan^2\alpha + \tan^2\beta) + 4k(1+k^2)\tan\alpha\tan\beta = (1-k^2)^2\tan^2\alpha\tan^2\beta,$$

and if $\tan\alpha$, $\tan\beta$ be the roots of the quadratic in m,

$$y = m(x+a) + \frac{a}{m} \quad \text{or} \quad m^2(x+a) - my + a = 0,$$

we get $\quad 4k^2\left\{y^2 - 2a(x+a)\right\} + 4k(1+k^2)\,a\,(x+a) = (1-k^2)^2 a^2$

or $$y^2 = -\frac{(1-k)^2}{k}a\left(x - \frac{(1-k)^2}{4k}a\right),$$
a parabola confocal with the original. It is obvious that we should get the same locus from the relation
$$\tan\tfrac{1}{2}a = -k\tan\tfrac{1}{2}\beta \text{ or } \tan\tfrac{1}{2}a\tan\tfrac{1}{2}\beta = k^{-1}.$$
When $k = -1$, $(\tan a - \tan\beta)^2 = 0$, the two tangents must coincide, and the locus reduces to the original $y^2 = 4a(x+a)$.

5917. (By W. H. H. Hudson, M.A.)—If the capital that is invested in machinery and raw material be proportional to the rate of interest, and if the proceeds of labourers' work after replacing capital be proportional to their wages and to the aforesaid capital conjointly, and if the employers, after paying wages and interest on the aforesaid capital, make the greatest possible profit; prove that the rate of interest in the pound must be equal to the ratio of the proceeds above mentioned to twice the aforesaid capital.

Solution by W. J. Macdonald, M.A.; Christine Ladd; *and others.*

Let C = capital, R = rate of interest per £1, I = interest on capital, P = proceeds, W = wages, G = profit; then we have
$$C = mR,\ P = nWC,\ G = P-(W+I),\ I = CR = mR^2 + P = mnWR.$$
Therefore $G = mnWR - W - mR^2 = \text{a max.}$; $\therefore\ \frac{dG}{dR} = mnW - 2mR = 0$;

therefore $nW = 2R$; therefore $R = \frac{nWC}{2C} = \frac{P}{2C}$.

6061. (By J. E. A. Steggall, M.A.)—If n letters are put at random into their n directed envelopes, find (1) the chance that no letter goes right; and (2) show that when n is infinite the chance is e^{-1}.

Solution by the Rev. W. Allen Whitworth, M.A.

The number of ways in which the letters can be put into the envelopes is $n!$ The number of ways so that all may be wrong is (*Choice and Chance*, Prop. xxxii.) $n!!$ Therefore the chance is
$$\frac{n!!}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \pm \frac{1}{n!}.\ \text{If } n = \infty \text{ the result becomes } \frac{1}{e}.$$

In *Choice and Chance*, 3rd Edition, p. 234, Ex. 233, I have pointed out that, if the number of letters exceed 8, the chance of all going wrong is

(correct to six decimal places) ·367879. Hence the chance of all going wrong is not *appreciably* affected by the number of letters. Whether there are 10 letters or 100 or 1000, the chance only differs from e^{-1} by a decimal figure of a very high order.

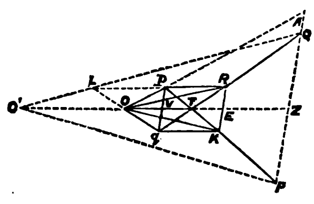
[The discussion of this famous historical problem began, so far as we know, with MONTMORT, in 1708, under the name of the game of *Treize*, and it has since been continued and extended by DE MOIVRE, EULER, LAPLACE, and others. Historical references are supplied in an editorial foot-note to Quest. 2397 (*Reprint*, Vol. VIII., p. 55), where solutions to another form of the Question are given by Messrs. WOOLHOUSE and WHITWORTH.

To suit our Printer's convenience, we use, here and everywhere, the symbol n ! for factorial n, and we have, in the foregoing solution, put n!! for what Mr. WHITWORTH calls sub-factorial n, for the definition whereof we refer to p. 98 of *Choice and Chance.*]

6108 & 6109. (By ELIZABETH BLACKWOOD and L. H. ROSENTHAL, B.A.)—P, Q, p, q are four points on a parabola, such that Pp and Qq are respectively parallel to the tangents at q and p. If O and O′ are the poles of pq and PQ respectively, and if O_{PQ} represents the perpendicular from O on PQ, &c., prove that (1) PQ = $3pq$, (2) $O_{PQ} = O'_{pq}$, (3) $O'_{PQ} = 9O'_{pq}$, (4) $O_{PQ} = 5O_{pq}$; (5) find the ratio of the areas of the triangles Opq, O′PQ; (6) find the locus of the intersections with PQ of tangents at p, q, (α) when pq passes through the focus, and (β) when pq is parallel to a fixed line.

Solutions by D. EDWARDES; J. O'REGAN; E. RUTTER; *and others.*

(1) Let R, K, V be middle points of Qq, Pp, pq. Then, since pR, OV, qK are all parallel to axis, we see, by construction, RT = qT; and therefore TQ = 3TR. Also, since pq, RK are evidently parallel, and R, K are middle points of qQ, pP; therefore RK, PQ are parallel, and therefore QP = 3RK = $3pq$.



(2) Since PQ, pq are parallel, and V is the middle point of pq, OV produced, bisects PQ in Z, and meets QL in O′, the pole of PQ. Hence we have

$$OO' = LR = 2pR = 2VE = VZ; \text{ therefore } O'V = OZ.$$

(3) $$OO' = 2pR = 2VE = 4OV;$$

therefore $$O'Z = OO' + OV + VZ = 2OO' + OV = 9OV.$$

(4) $$OZ = OV + VZ = OV' + OO' = 5OV.$$

(5) Porduce qO to meet Rp in L; then, clearly, QL is tangent at Q, and is parallel to OR, and similarly tangent at P is parallel to OK. But

QP = 3RK; therefore

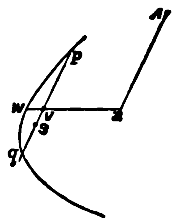
$$\Delta O'PQ = 9\Delta ORK = 9(\Delta ORT + \Delta OKT + \Delta RTK)$$
$$= 9(\Delta RpT + \Delta TqK + \Delta RTK) = 27\Delta pTq = 27\Delta Opq.$$

(6) Let A be intersection of Op, PQ. Then $Ap = qQ = 4qT = 4Op$. Hence, if x', y' be coordinates of A, and x, y those of p, we easily get

$$x' = 4a + 5x, \quad y = \frac{12ax' - 8a^2}{5y'};$$

whence equation to locus is $5y'^2(x'-4a) = 4a(3x'-2a)^2$.

If pq coincide with latus rectum, we get $x' = 9a$, $y' = 10a$, which satisfy the above equation. In the second case, OZ is a fixed diameter, and if W be middle point of OV, we readily find $AZ = 5pV$, $WZ = 9WV$. Hence $AZ^2 = \frac{25}{9}m \, . \, ZW$, where m is a constant. Hence the locus in this case is a similarly situated parabola.



We can find any number of points on the locus in the first case as follows:—

Draw any focal chord pSq. Draw the diameter WV, and make $VZ = 8VW$. Through Z draw ZA parallel to pSq, and make $AZ = 10VW$, then A is a point on the locus.

6009. (By the late Professor Clifford, F.R.S.)—Find the form of a kite-string under the action of the wind.

Solution by W. J. Curran Sharp, M.A.

Let the point of attachment be taken as origin, the axis of x horizontal in the direction of the wind, and the axis of y vertical, and let μ denote the pressure of the wind on a vertical unit of the string, and m its mass; then the equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + \mu\frac{dy}{ds} = 0 \text{ and } \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mg = 0;$$

therefore $$T\frac{dx}{ds} + \mu y = a \text{ and } T\frac{dy}{ds} + mgs = b,$$

(where $a = P\cos\alpha$, $b = P\sin\alpha$, P being the pull at the point of attachment, and α its inclination to horizon); hence the differential equation to the curve required is

$$\frac{dy}{dx} = \frac{mgs-b}{\mu y - a} \text{ or } (\mu y - a)\frac{d^2y}{dx^2} + \mu\left(\frac{dy}{dx}\right)^2 = mg\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}};$$

hence we have $$\rho = \frac{(\mu y - a)\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{mg\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} - \mu\left(\frac{dy}{dx}\right)^2}.$$

Thus, at the point whose height is h, and where the inclination of the string to the horizon is θ, $$\rho = \frac{(\mu h + a)}{mg\cos^2\theta - \mu\sin^2\theta\cos\theta}.$$

So that, for an inflexion, $\cos\theta = 0$ or $mg\cos\theta = \mu\sin^2\theta$.

[illegible]

... where C must be determined from the initial conditions.

6206. (By H. McColl, B.A.)—Given (1) that, whenever the statements a, b, c are either all three true or all three false, then the statement x is false and y true, or else x is true and y false; (2) that whenever d, e, y are either all three true or all three false, then the statement a is false and x true, or a is true and x false. When may we infer from these premises that either c or y is true?

I. *Solution by* C. J. MONRO, M.A.

In BOOLE's forms, the question may be stated as follows :—

If $\quad (abx + a'b'x')(cy + c'y') = 0, \quad (ax + a'x')(dey + d'e'y') = 0$ ......(1, 2), when is $x'y' = 0$? Clearly, from (1), when $a'b'c' = 1$; and, from (2), when $a'd'e' = 1$; that is, when a, b, and c, or a, d, and e, or all, are false.

To see whether otherwise, make $x'y' = t'$; then (IX. 9) $Et + Ft' = 0$, where F is $a'b'c' + a'd'e'$, the coefficient of $x'y'$ in the sum of (1) and (2), and E is the product of the coefficients of xy, xy', $x'y$. As E contains aa', $Ft' = 0$; which means that the truth of $a'b'c'$, or $a'd'e'$, or (VI. 5, IX. 2) both, implies the truth of x or y, or both. So the above answer exhausts the data.

II. *Solution by* ELIZABETH BLACKWOOD.

Adopting the Proposer's method and notation, the data are expressed by the complex statement

$$(abx + a'b'x' : c'y + cy')(dey + d'e'y' : a'x + ax');$$

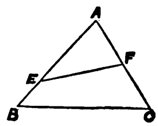
hence $\quad xy' : (a+b+c)(d+e+a)$, $\quad$ that is, $\quad x'y' : a + (b+c)(d+e)$;

hence, by transposition (*i.e.* contra-position) $\quad a'(b'c' + d'e') : x + y$.

The above solution is exhaustive, and contains every step of the working necessary for obtaining the required antecedent of $x + y$.

6143. (By C. B. S. CAVALLIN, M.A.)—Find the average area of the triangle cut off by a random line from an equilateral triangle.

Solution by Prof. MATZ; Prof. SEITZ; *and others.*



Let ABC be the equilateral triangle, EF the random line. Owing to the regularity of the triangle, it is necessary to consider only those positions of the random line in which it cuts AB and AC, and the angle AEF is less than the angle B.

Let $AB = a$, $AE = x$, $\angle AEF = \theta$, and area $AEF = u$.

Then we have $u = \dfrac{x^2\sqrt{3}\sin\theta}{4\sin(\frac{1}{3}\pi + \theta)}$; and the required average is

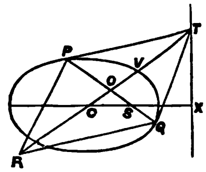
$$\Delta = \frac{\int_0^{\frac{2}{3}\pi}\int_0^a u\sin\theta\, d\theta\, dx}{\int_0^{\frac{2}{3}\pi}\int_0^a \sin\theta\, d\theta\, dx} = \frac{\sqrt{3}}{2a}\int_0^{\frac{2}{3}\pi}\int_0^a \frac{x^2\sin^2\theta\, d\theta\, dx}{\sin(\frac{1}{3}\pi + \theta)}$$

$$= \frac{a^2\sqrt{3}}{6}\int_0^{\frac{2}{3}\pi} \frac{\sin^2\theta\, d\theta}{\sin(\frac{1}{3}\pi + \theta)} = \frac{a^2\sqrt{3}}{24}(3\log 3 - 2).$$

6115. (By A. J. C. ALLEN, B.A.)—TP, TQ are tangents at the ends of a focal chord PSQ of an ellipse, and the parallelogram TPRQ is completed; find the locus of R.

Solution by G. TURRIFF, M.A.; G. HEPPEL, M.A.; *and others.*

The tangents at P and Q intersect in the directrix; and, putting $\angle$ RCX $= \theta$, we have



$$CT \, . \, CO = CV^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta};$$

also $\qquad CT - CR = 2CO;$

therefore $\quad CT^2 - CT \, . \, CR = 2CT \, . \, CO,$

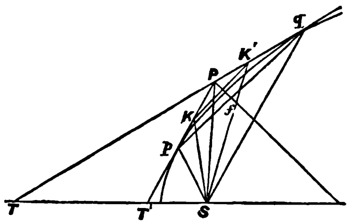
and $\qquad CT = -\dfrac{a}{e\cos\theta}.$

Hence, if $CR = \rho$, we get

$$\frac{a^2}{e^2\cos^2\theta} + \frac{a\rho}{e\cos\theta} = \frac{2a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}; \quad (a+ex)(b^2x^2+a^2y^2) = 2ab^2e^2x^2.$$

6233. (By C. MORGAN, B.A.)—Pp, Pq are tangents to a parabola, and the bisectors of the angles pPq, PSp, PSq cut pq, Pp, Pq in f, K, K′ respectively; prove that Pf, KK′ intersect at right angles.

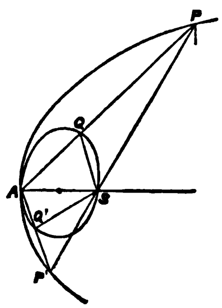
Solution by A. ANDERSON, B.A.; J. S. JENKINS, L.C.P.; *and others.*



Now $\angle$ KSK′ (= PSp) = TPT′,
where $\angle$ KSK′ + KPK
= TPT′ + KPK′ = 2 rt. angles;
therefore S, K, P, K′ lie on a circle, and $\angle$ PK′K = PKK′
(since PSK = PSK′);
therefore PK′ = PK;
and since $\angle$ KPK′ is bisected by Pf, and $\angle$ PK′K = $\angle$ PKK′,
therefore Pf intersects KK′ at right angles.

6239. (By T. MORLEY, L.C.P.)—The minor axis SA of an ellipse is equal to the distance between the foci; PSP′ is a focal chord of a parabola, whose focus is S and vertex A; AP, AP′ cut the ellipse in Q, Q; prove that QSQ′ is a right angle.

Solution by R. KNOWLES, B.A., L.C.P.; C. H. SAMPSON; *and others.*



Let SQ bisect the angle ASP; then, if

$SQ = \rho, \quad SP = r, \quad SA = a, \quad \angle PSA = 2\phi,$

we have $\rho(a+r)\sin\phi = ar\sin 2\phi,$

or $$\frac{2\cos\phi}{\rho} = \frac{1}{a}+\frac{1}{r} = \frac{1}{a}+\frac{1+\cos 2\phi}{2a} = \frac{1+\cos^2\phi}{a},$$

and the locus of Q is $2ax = 2x^2+y^2$,

which represents an ellipse whose minor axis is SA, equal to the focal distance; therefore

$$\angle QSQ' = \tfrac{1}{2}PSP' = \tfrac{1}{2}\pi.$$

5763. (By Professor SYLVESTER, F.R.S.)—Prove that

$$\Sigma_{i=0}^{i=\infty} \frac{(-)^i x^{\frac{1}{2}(i^2+i)}}{(1-x^k)(1-x^{k+1})\,\ldots\,(1-x^{k+i})},$$

when k is any positive integer, is a rational integral function of the degree k^2-k in x. Thus, for example,

$$\frac{1}{1-x}-\frac{x}{(1-x)(1-x^2)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}\ldots\ldots = 1,$$

$$\frac{1}{1-x^2}-\frac{x}{(1-x^2)(1-x^3)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}\ldots\ldots = 1-x+x^2.$$

Solution by W. J. CURRAN SHARP, M.A.

Let u_k denote $\Sigma_{i=0}^{i=\infty} \dfrac{(-)^i x^{\frac{1}{2}(i^2+i)}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})}$; then

$$u_1 = \frac{1}{1-x}-\frac{x}{(1-x)(1-x^2)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{-y}{1-y}-\frac{y^2}{(1-y)(1-y^2)}-\frac{y^3}{(1-y)(1-y^2)(1-y^3)}, \text{ \&c., if } y = \frac{1}{x}$$

$$= 1+\frac{1}{1-y}+\frac{1}{(1-y)(1-y^2)}+\text{\&c.}$$

$$-\frac{1}{1-y}-\frac{1}{(1-y)(1-y^2)}-\frac{1}{(1-y)(1-y^2)(1-y^3)}-\text{\&c.} = 1;$$

$$\frac{u_{k+1}}{1-x^k} = \frac{1}{(1-x^k)(1-x^{k+1})}+\ldots+\frac{(-)^{i-1}x^{\frac{1}{2}(i^2-i)}}{(1-x^k)(1-x^{k+i})\ldots(1-x^{k+i})}+\text{\&c.},$$

$$x^k u_k = \frac{x^k}{1-x^k}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{(-)^i x^{\frac{1}{2}(i^2+i)+k}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})}+\text{\&c.},$$

therefore $\quad u_{k+1}(1-x^k)^{-1}+x^k u_k = x^k(1-x^k)^{-1}+u_k,$

therefore $\quad u_{k+1} = (1-x^k)^2 u_k + x^k.$

Therefore, since u_1 is of the 0th degree, u_2 will be an integral function of the 2nd degree, u_3 an integral function of 6th degree, and u_k an integral function of the $0+2+4+6+\dots+2(k-1) = (k^2-k)$th degree in x.

5764. (By Dr. HIRST, F.R.S.)—The projections, from a centre A, upon a fixed axis a, of n given lines $a_1, a_2 \dots a_n$ in space, form a range which is homographic with a given range $A_1, A_2 \dots A_n$. Prove that (1) when $n = 4$, the locus of A is a quartic surface having a for a double line, and passing through each of the four given lines, as well as through each of the eight lines which can be drawn to intersect three of the latter and the axis a; (2) when $n = 5$, the locus of A is a curve of the seventh order, which cuts each of the five given lines four times, and the axis a six times; and (3) when $n = 6$, there are four positions of the centre A.

Solution by J. J. WALKER, M.A.

1. Taking a as axis of x', let $a_1 \dots a_4$ be determined by the planes

$$l'_1 \equiv x' + B_1 y' + C_1 z' = x' + n'_1 = 0, \text{ and } m'_1 \equiv B'_1 y' + C'_1 z' + 1 = 0 \dots,$$

the centre A being (x, y, z). The plane through A and a_1 cuts a in the point

$$x'_1 = l_1 : m_1, \text{ where } l_1 \equiv x + B_1 y + C_1 z = x + n_1,\ m_1 \equiv B'_1 y + C'_1 z + 1.$$

The locus of A is, therefore,

$$(l_1 m_2 - l_2 m_1)(l_3 m_4 - l_4 m_3) + k'(l_1 m_3 - l_3 m_1)(l_2 m_4 - l_4 m_2) \dots\dots\dots (1),$$

which is of the form $\quad u_2 x^2 + (u_3 + u'_2)x + u_4 + u'_3 + u''_2 = 0 \dots\dots\dots (2),$

the suffixes indicating the orders of homogeneous functions of y, z. The form (2) shows that a is a double line on the quartic surface, locus of A; and the form (1) that the surface passes through each of the four lines $a_1 \dots a_4$. Either of the lines meeting a and any triad of these four will meet the surface in *five* points, since it meets the surface twice in a, and must therefore lie wholly in the surface.

2. When $n = 5$ the locus of A is the curve of intersection of (1), with

$$(l_1 m_2 - l_2 m_1)(l_3 m_5 - l_5 m_3) + k(l_1 m_3 - l_3 m_1)(l_2 m_5 - l_5 m_2) = 0,$$

or $\quad v_2 x^2 + (v_3 + v'_2)x + v_4 + v'_3 + v''_2 = 0 \dots\dots\dots (3).$

These two quartic surfaces have in common the double line a (counting, therefore, as *four*), the lines a_1, a_2, a_3, and the pair meeting them and a. Their curve of intersection can therefore be met by a plane in only $16-4-3-2$, or 7 points. The line a_4 lying on the surface of (1) is cut by the surface (3) in four points, which must be points in this curve of order 7; and similarly a_5, and by symmetry the other lines of the five. If x is eliminated from (2) and (3), the result is $u_{12} \dots + u_8 = 0$,—the degree of which is accounted for by the fourfold line a not being included,—a curve passing eight times through the origin, two of which intersections are accounted for by the lines meeting a and each of the triad $a_1 a_2 a_3$. The remaining six passages through the origin indicate so many intersections

of a by the curve, locus of A. Hence, any plane through a determines a single point on that locus. In fact, such a plane cuts the surfaces (2), (3) in two conics, three of the points common to which are those in which the plane is met by $a_1 a_2 a_3$, while the fourth is the single point on the curve, locus of A, determined by the plane.

3. If therefore, lastly, $n = 6$, the positions of the centre A will be any points common to the curves of intersection of (2), (3), and of the former with a third quartic surface

$$(l_1 m_2)(l_3 m_6) + k''(l_1 m_3)(l_2 m_6) = 0, \text{ or } w_2 x^2 + \ldots + w''_2 = 0 \ldots\ldots(4),$$

and, by what has been remarked above, these points may be conveniently determined by planes through a. Substituting λy for z in the equations (2), (3), (4), and discarding the factor y^2, the condition that the three conics—which are known to have three points in common, viz., those in which the plane $\lambda y - z$ meets the triad of lines $a_1 a_2 a_3$—shall have a fourth common point of intersection, gives, on restoring $z : y$ for λ,

$$k(124)(356) + k'(125)(364) + k''(126)(345) + k'k''(134)(256)$$
$$+ k''k(135)(264) + kk'(136)(245) = 0,$$

a homogeneous function of the fourth order in yz............(5),

where (124) stands for the determinant $\begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix}$; and the other symbols for similar determinants. Hence there are four positions of the centre A, determined by the four planes through a which form the locus of the above equation.

It may be verified *à posteriori*—what is evident from geometrical considerations—that no one of the equations (1), (3), (4), (5) is altered by changing k into $k+1\ldots$, while interchanging 1 and 3 in the suffixes; or into $1 : k\ldots$, interchanging 2 and 3; or into $k : 1+k\ldots$, interchanging 1 and 2.

5991. (By the Editor.)—A wire hoop is cut at random into three pieces; find the respective probabilities that the three pieces will, when straightened, admit of being formed into (1) a triangle of any kind, (2) an acute-angled triangle.

Solution by C. J. Monro, M.A.

The only conditions are that the three quantities are positive and their sum constant. The same is true of the trilinear coordinates of a point inside an equiangular triangle of reference; and, since the probabilities of range in the one case, and the area in the other, vary merely as the product of any two of the three differentials, the values required will be those of certain areas as fractions of the area of the triangle.

Let x, y, z be the coordinates, then the areas will be bounded by the lines given by unlike signs in

$$x \pm y \pm z = 0, \quad x^2 \pm y^2 \pm z^2 = 0 \ldots\ldots\ldots\ldots(1, 2);$$

and since the three pieces may be equal, the area will be the central one in each case.

ABC being the triangle, equations (1) give
another triangle abc, with corners bisecting the
sides of ABC, and its area gives the chance $\frac{1}{4}$.

To interpret (2), complete the parallelogram
CAC′B. C′A, C′B are asymptotes to $x^2+y^2=z^2$,
as is evident on writing this

$$2(y+z)(z+x)=(x+y+z)^2;$$

for $y+z$, $z+x$ represent C′A, C′B, and $x+y+z$
the line infinity. The curve obviously bisects the
the two others forms a tricuspid figure; the arc
course, with finite curvature. Now aB, and bc p
BC′ in a', are ordinates to an asymptote; so we hav
one curved side, $=AC'a'b\,(\log C'B-\log C'a')$, $=\Delta$ l
of the triangle of reference. Therefore Cab (curved)
the tricuspid area, as a fraction of Δ, is $3\log 2-$
tained by the Editor (*Reprint*, Vol. II., p. 74)
(Vol. V., p. 33) in solutions of Question 1100. In fa
might be considered a re-interpretation of the Editor
BCba' being a parallel-projection of his ABDC. S
83–87) Prof. Clifford's paper on the probability of s
make a polygon of n sides.

The advantage of the present enunciation is that, w
see it to be mathematically equivalent to that of Qu
poses the division of a finite straight line, it leaves no
segments are symmetrically related in respect of the
lengths. But it may be worth while to show dir
thought to have been too implicitly assumed, that th
of the finite line. In a line of unit length, $2(1-u)\,du$
the length of a specified segment is between u and $u+d$
middle one, $1-u$ is the chance that the first cut leaves
at such a distance to the right, and the like to the left,
together; and du is the chance that the second is at
either case. As to an end segment, $2du$ is the chance th
cut is made at a distance between u and $u+du$ from a
$1-u$ that the other is made further off.

6252. (By Dr. Hopkinson, F.R.S.)—It is found that
loses 0·000001 of its charge per second by conduction throu
The specific inductive capacity of the glass of which it is mad
that the resistance of a cubic centimetre of the glass is rough
ohms, having given that one electro-magnetic unit of quantity
3×10^{10} electro-static units, and that one ohm is equal to 10
magnetic units.

Solution by Professor G. Carey Foster, F.R.S.

Let R = resistance of jar to conduction between coatings, ρ =
resistance of glass, S = electrostatic capacity of jar, σ = specific indu

$$\frac{T}{p^2} - \frac{dT}{d\phi} \cot \alpha = T; \text{ whence}$$

the equation $z^2 - z \cot \alpha = 1$,

$$\frac{\ldots + A_2 \epsilon^{m_2\phi}}{\ldots + m_2 A_2 \epsilon^{m_2\phi}},$$

$(1 - m_2 \tan \theta)$;

$$\frac{\tan Q}{\ldots \theta - 1};$$

$$\frac{A_1}{A_2} \left(\frac{a}{c}\right)^{m_1 - m_2 \tan \alpha} = -1,$$

$$AP = a \left(\frac{m_2 \tan \theta - 1}{m_1 \tan \theta - 1}\right)^{\frac{\cot \alpha}{m_1 - m_2}};$$

; arc $AP = a \left(\frac{\tan \theta + 2}{2 - 4 \tan \theta}\right)$.

$$\left(\frac{m + \tan \theta}{m - m^2 \tan \theta}\right)^{\frac{m^2-1}{m^2-1}},$$

ic equation of the spiral is

ot ϕ;

the pole (not from the point (or AP) is measured from the n, $s = 0$ when $\phi = -\infty$.

—Solve the equation

$c^2ax + c^2b = 0$.

SER.

$B) = bc^{-1}$; then $Au^6 + Bv^6 = 0$.

wn.

hin the perimeter ($= s$) of an ius r, three points are taken the circle through these three s $\frac{3}{5} \pi^2 r^2 s^{-2}$; and (2) when the is then that of the EDITOR's

4549, 4566, 4578, 4579. (By C. LEUDESDORF, M.A., and E. B. ELLIOTT, M.A.)—A bag contains mn balls, each equally likely to be of any one of m different colours. All but m are drawn out at random, and found to be $n-1$ of each colour. Find the chance that the remaining m are (1) one of each colour, (2) all of one colour.

I. *Solution by* T. J. SANDERSON, M.A.; E. B. ELLIOTT, M.A.; *and others.*

There is no presumption in favour of any one colour, that is, all colours are equally likely for each of the remaining m balls. Hence the number of all possible changes that may be presented is m^m. Now, the number of ways in which m different colours may be presented, = number of permutations of m things, is $m\,!$; hence the chance that we find one of each colour is $\dfrac{m\,!}{m^m}$. Again, the number of ways in which all can be of one is m; hence the chance that all are of one colour is $\dfrac{m}{m^m} = m^{1-m}$.

The solutions to Questions 4549 ($m=3$), 4566 ($m=4$), 4579 ($m=5$) follow at once, and give, for the two cases, the following results:—

$$\frac{3\,!}{3^3}=\frac{2}{9},\ \frac{3}{3^3}=\frac{1}{9};\quad \frac{4\,!}{4^4}=\frac{3}{32},\ \frac{4}{4^4}=\frac{1}{64};\quad \frac{5\,!}{5^5}=\frac{24}{625},\ \frac{5}{5^5}=\frac{1}{625}.$$

II. *Solution by* C. LEUDESDORF, M.A.

1. Take first the simpler case where $m = 3$ (Quest. 4549). The 10 possible hypotheses may be grouped together under three heads:—

$$(a)\ (n, n, n),\quad (b)\ (n+1, n, n-1),\quad (c)\ (n+2, n-1, n-1);$$

i.e., there may be n balls of each colour; or $n+1$ of one colour, n of second, $n-1$ of the third, &c. Only one hypothesis is of the type (a); six fall under the head (b), and three under (c). The chances of drawing $n-1$ balls of each colour, supposing (a), (b), (c) in turn to be true, are

$$\frac{6\,(n\,!)^3\,\overline{3n-3}\,!}{(\overline{n-1}\,!)^3\,3n\,!},\quad \frac{\overline{3n+1}\,!\,n\,!\,\overline{n-1}\,!\,\overline{3n-3}\,!}{(\overline{n-1}\,!)^3\,3n\,!},\quad \frac{(\overline{n-1}\,!)^2\,\overline{n+2}\,!\,\overline{3n-3}\,!}{(\overline{n-1}\,!)^3\,3n\,!};$$

$$\text{or}\quad \frac{6n^3}{N},\ \frac{3n^2\,(n+1)}{N},\ \frac{n\,(n+1)\,(n+2)}{N},\qquad \text{where } N = 3n\,(3n-1)\,(3n-2).$$

If we multiply these by the factors 1, 6, 3 respectively, and add, the result is $\dfrac{3n\,(3n+1)\,(3n+2)}{N}$; so that the required chances are

$$(1)\ \frac{6n^3}{3n\,(3n+1)\,(3n+2)}=\frac{n^3}{C_3^{3n+2}},\quad \text{and}\quad (2)\ \frac{n\,(n+1)\,(n+2)}{3n\,(3n+1)\,(3n+2)}=\frac{C_3^{n+2}}{C_3^{3n+2}}.$$

2. Proceeding now to the general case, it is seen that the number of possible hypotheses may be very large; but as we are only concerned with the first and the last of these—those, namely, in which the balls are supposed to have been (1) n of each colour, and (2) $n-1$ of all the colours but one, of which colour are the $n+m-1$ remaining balls; and as the sum of the chances, supposing each hypothesis in turn true, must be unity, we may write down the chances required precisely as in the particular case.

capacity of glass, and $V =$ difference of potentials between coatings: then we have $\frac{dQ}{dt} = \frac{V}{R} = \frac{Q}{RS} = \frac{4\pi Q}{\rho\sigma}$, since $\rho\sigma = 4\pi RS$;

that is, $\rho = \frac{4\pi Q}{\sigma} \cdot \frac{dt}{dQ}$, or, with the given values, $\rho = \frac{1}{2}\pi \times 10^6$ in electro-static measure.

To reduce to ohms, multiply by $(3 \times 10^{10})^2$, and divide by 10^9; then

$$\rho = \tfrac{1}{2}\pi \times 9 \times 10^6 \times 10^{20} \times 10^{-9} = 14{\cdot}1\ldots \times 10^{17}.$$

6253. (By B. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle.

Solution by Professor Townsend, F.R.S.

This pretty property follows immediately from two others, both of which are well-known,—viz., (1) That when, of a figure of invariable form moving in a plane, two of the lines envelop any curves in the plane, the several lines all envelop corresponding curves in the plane, the corresponding normals to which all pass through a common point in the plane; and (2) That when, of a figure of invariable form moving in a plane, two of the lines envelop circles in the plane, the several lines all envelop circles in the plane, the centres of which all lie on a common circle with the intersection of the corresponding radii in the plane. From these combined, the property in question is an obvious and immediate consequence.

In the first of the above properties, the common point of concurrence of the several normals, in any position of the figure, is obviously the momentary centre of motion, corresponding to the position of the figure in the plane; and in the second, the common circle of position of the several centres, throughout the entire motion, is obviously that passing through those of the two given circles and through the point of concurrence of the parallels through them to the two given lines, in any position of the figure in the plane.

6224. (By Professor Morel.)—Eliminer ϕ et ψ de l'équation

$$Z = x\phi(z) + y\psi(z).$$

Solution by R. Tucker, M.A.; Prof. Scott, M.A.; *and others.*

$$\frac{dZ}{dx} = \phi + x\phi' + y\psi', \quad \frac{dZ}{dy} = x\phi' + \psi + y\psi', \quad \text{therefore } \frac{dZ}{dx} - \frac{dZ}{dy} = \phi - \psi;$$

$$\text{therefore } \frac{d^2Z}{dx^2} - \frac{d^2Z}{dx\,dy} = \phi' - \psi' = \frac{d^2Z}{dx\,dy} - \frac{d^2Z}{dy^2}, \quad i.e., \frac{d^2Z}{dx^2} - 2\frac{d^2Z}{dx\,dy} + \frac{d^2Z}{dy^2} = 0.$$

$$S_{3,1} = \frac{n+2\,!}{n-1\,!\;3\,!}\cdot\frac{3\,!}{1\,!\;2\,!} = \tfrac{1}{2}n\,(n+1)\,(n+2),$$

$$S_{3,2} = 2\,\frac{n\,!\;n+1\,!}{(n-1\,!)^2\,1\,!\,2\,!}\cdot\frac{3\,!}{2\,!\;1\,!} = 3n^2\,(n+1),$$

$$S_{3,3} = \frac{n\,!\,n\,!\,n\,!}{(n-1\,!)^3\,1\,!\,1\,!\,1\,!}\cdot\frac{3\,!}{3\,!\,0\,!} = n^3,$$

$$S_{3,1} + \&c. = \tfrac{1}{2}n\,(3n+1)\,(3n+2),$$

$$P_{3,1} = \frac{(n+1)\,(n+2)}{(3n+1)\,(3n+2)},\qquad P_{3,3} = \frac{2n^2}{(3n+1)\,(3n+2)};$$

and the chances are (1) $\frac{1}{9}P_{3,3}$, (2) $\frac{2}{9}P_{3,1}$.

Let $m = 4$; then we can take $n_1 = 4$, $n_1+n_2 = 4$, $n_1+n_2+n_3 = 4$, $n_1+n_2+n_3+n_4 = 4$, giving

$$S_{4,1} = \frac{n+3\,!}{n-1\,!\,4\,!}\cdot\frac{4\,!}{1\,!\,3\,!} = n\,C_3^{n+3},$$

$$S_{4,2} = \left[2\,\frac{n\,!\;n+2\,!}{(n-1\,!)^2\,1\,!\,3\,!} + \frac{n+1\,!\;n+1\,!}{(n-1\,!)^2\,2\,!\,2\,!}\right]\frac{4\,!}{2\,!\,2\,!} = \tfrac{1}{2}n^2\,(n+1)\,(7n+11),$$

$$S_{4,3} = 3\,\frac{n\,!\,n\,!\,n+1\,!}{(n-1\,!)^3\,1\,!\,1\,!\,2\,!}\cdot\frac{4\,!}{3\,!\,1\,!} = 6n^3\,(n+1),\qquad S_{4,4} = n^4,$$

$$S_{4,1} + \&c. = \tfrac{1}{3}\,(32n^4 + 48n^3 + 22n^2 + 3n),$$

$$P_{4,1} = \tfrac{1}{2}\,\frac{(n+1)\,(n+2)\,(n+3)}{32n^3+48n^2+22n+3},\quad P_{4,4} = \frac{3n^3}{32n^3+48n^2+22n+3};$$

and the chances are (1) $\frac{1}{64}P_{4,4}$, (2) $\frac{5}{32}P_{4,1}$.

Let $m = 5$, then we can take $n_1 = 5$, $n_1+n_2 = 5$, $n_1+n_2+n_3 = 5$, $n_1+n_2+n_3+n_4 = 5$, $n_1+n_2+n_3+n_4+n_5 = 5$, giving

$$S_{5,1} = \frac{n+4\,!}{n-1\,!\,5\,!}\cdot\frac{5\,!}{1\,!\,4\,!} = n\,C_4^{n+4},$$

$$S_{5,2} = \left[2\frac{n\,!\;n+3\,!}{(n-1\,!)^2\,1\,!\,4\,!} + 2\frac{n+1\,!\;n+2\,!}{(n-1\,!)^2\,2\,!\,3\,!}\right]\frac{5\,!}{2\,!\,3\,!} = \tfrac{5}{6}n^2(n+1)(n+2)(3n+5),$$

$$S_{5,3} = \left[3\,\frac{n\,!\,n\,!\,n+2\,!}{(n-1\,!)^3\,1\,!\,1\,!\,3\,!} + 3\,\frac{n\,!\,n+1\,!\,n+1\,!}{(n-1\,!)^3\,1\,!\,2\,!\,2\,!}\right]\frac{5\,!}{3\,!\,2\,!}$$
$$= \tfrac{5}{2}n^3\,(n+1)\,(5n+7),$$

$$S_{5,4} = 4\,\frac{n\,!\,n\,!\,n\,!\,n+1\,!}{(n-1\,!)^4\,1\,!\,1\,!\,1\,!\,2\,!}\cdot\frac{5\,!}{4\,!\,1\,!} = 10n^4\,(n+1),$$

$$S_{5,5} = n^5,\quad S_{5,1} = \tfrac{1}{24}n\,(5n+1)\,(5n+2)\,(5n+3)\,(5n+4) = \tfrac{1}{24}n\,C_4^{5n+4},$$

$$P_{5,1} = \frac{C_4^{n+4}}{C_4^{5n+4}},\qquad P_{5,5} = \frac{n^4}{C_4^{5n+4}};$$

and the chances are (1) $\frac{1}{625}P_{5,5}$, (2) $\frac{24}{625}P_{5,1}$.

[Mr. TEBAY is of opinion that "there is no ambiguity in the statement of the problem," but that "the first solution lacks the *à priori* consideration upon which the conclusion depends, and that the second solution is too direct, and lacks the conclusion."]

6181. (By DONALD MCALISTER, D.Sc.)—Prove that the nth odd power of an odd number greater than unity can be presented as the difference of two square whole numbers in n different ways.

Solution by Prof. SCOTT, M.A.; G. EASTWOOD, M.A.; *and others.*

Take the odd number N; then $\frac{1}{2}(N \pm 1)$, $\frac{1}{2}(N^2 \pm 1)$, ..., are all integers. The nth odd power is

$$\begin{aligned} N^{2n-1} &= N^{2n-2}\left\{\tfrac{1}{2}(N+1)\right\}^2 - N^{2n-2}\left\{\tfrac{1}{2}(N-1)\right\}^2 \\ &= N^{2n-4}\left\{\tfrac{1}{2}(N^3+1)\right\}^2 - N^{2n-4}\left\{\tfrac{1}{2}(N^3-1)\right\}^2 \\ &= \dots = \left\{\tfrac{1}{2}(N^{2n-1}+1)\right\}^2 - \left\{\tfrac{1}{2}(N^{2n-1}-1)\right\}^2. \end{aligned}$$

[$N^{2n-1} = y^2 - z^2$ is satisfied in whole numbers, provided that $y - z = N^r$, $y + z = N^{2n-r-1}$, if N, and therefore $N^r \cdot N^{2n-r-1}$, be an odd number; and r may be any whole number from 0 to $n-1$, giving n solutions. If $r > n-1$, we only get z negative, and the values of y^2, z^2 are repeated.]

6097. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_0^\infty \frac{\log x}{(1+x)^4}\,dx = -\tfrac{1}{3}, \quad \int_0^\infty \frac{1-3x}{(1+x)^5}(\log x)^2\,dx = 1,$$

$$\int_0^\infty \frac{1-10x}{(1+x)^6}(\log x)^3\,dx = -3, \quad \int_0^\infty \frac{1-25x+40x^2}{(1+x)^7}(\log x)^4\,dx = 12, \text{ \&c.};$$

$$U_{n-2} = \int_0^\infty \frac{a_0 - a_1 x + a_2 x^2 - \dots}{(1+x)^{n+1}}(\log x)^{n-2}\,dx = (-1)^n \frac{(n-2)!}{2}$$

f $a_0 = 1$, $a_1 = 2^{n-1} - (n+1)$, $a_2 = 3^{n-1} - (n+1)\,2^{n-1} + \frac{1}{2}(n+1)\,n$, and so on; the number of such coefficients being $\frac{1}{2}(n-1)$, or $\frac{1}{2}n$, as n is odd or even.

I. *Solution by* H. STABENOW, M.A.; T. R. TERRY, M.A.; *and others.*

The coefficients in any three of these integrals, as U_{2n-4}, U_{2n-3}, U_{2n-2}, say $b_0, b_1, \dots, b_k, \dots, b_p, \dots, b_m, \dots, b_{n-2}$; $c_0, c_1, \dots, c_{n-2}$; $d_0, d_1, d_2, \dots, d_{n-1}$; are connected by the following relations:—

$$c_{p+1} = (p+2)\,b_{p+1} - (p+1)\,b_p + (2n-1)\,b_p \quad \dots\dots\dots\dots\dots(1),$$

$$d_{p+1} = (p+2)\,c_{p+1} - (p+1)\,c_p + (2n)\,c_p \quad \dots\dots\dots\dots\dots\dots(2),$$

$$\text{with } p = 0 = 1 = 2 = \dots = (n-2),$$

which are verified by extending decomposition and reduction—as of $m \cdot p^k$ into $p^{k+1} + (m-p)\,p^k$ and of $(m)_{k+1}p^k + (m)_k p^k$ to $(m+1)_{k+1}p^k$—to all terms that contain the factor $(p+1)$ or $(p+2)$, and observing that

$$\begin{aligned} d_{n-1} &= n^{2n-2} - (2n)_1(n-1)^{2n-2} + (2n)_2(n-2)^{2n-2} - \dots \mp (2n)_{n-2}\,2^{2n-2} \pm (2n)_{n-1} \\ &= \tfrac{1}{2}\,[x = -n \text{ and } \Delta x = 1]\,\left\{\Delta^{2n} \cdot x^{2n-2}\right\} = 0, \quad \pm n c_{n-1} = 0 \dots\dots\dots(3). \end{aligned}$$

With the help of (1), (2), (3) we deduce, integrating by parts,

$$U_{2n-4}=\frac{1}{2n-3}\int_0^\infty\left\{\frac{x\,(b_0-b_1x+\ldots\pm b_{n-2}x^{n-2})}{(1+x)^{2n-1}}(\log x)^{2n-3}\right\}$$

$$-\frac{1}{2n-3}\int_0^\infty(\log x)^{2n-3}\,d_x\frac{x\,(b_0-b_1x+\ldots\pm b_{n-2}x^{n-2})}{(1+x)^{2n-1}}$$

$$=-\frac{1}{2n-3}\int_0^\infty\frac{(c_0-c_1x+\ldots\pm c_{n-2}x^{n-2})}{(1+x)^{2n}}(\log x)^{2n-3}\,dx$$

$$\pm\frac{nb_{n-2}}{2n-3}\left\{\int_0^\infty\frac{x^{n-1}}{(1+x)^{2n}}(\log x)^{2n-3}\,dx\right\}=\dot{\mathrm{J}};$$

but, substituting $(y^{-1}-1)$ for x, and $(1-z)$ for y, we find

$$\dot{\mathrm{J}}=\int_0^{\frac{1}{2}}(1-y)^{n-1}y^{n-1}\left[\log\left(\frac{1}{y}-1\right)\right]^{2n-3}dy$$

$$+\left\{\int_{\frac{1}{2}}^{+1}(1-y)^{n-1}y^{n-1}\left[\log\left(\frac{1}{y}-1\right)\right]^{2n-3}dy\right\}=\dot{\mathrm{J}}_1,$$

and $$\dot{\mathrm{J}}_1=-\int_0^{\frac{1}{2}}(1-z)^{n-1}z^{n-1}\left[\log\left(\frac{1}{z}-1\right)\right]^{2n-3}dz$$

$$=-\int_0^{\frac{1}{2}}(1-y)^{n-1}y^{n-1}\left[\log\left(\frac{1}{y}-1\right)\right]^{2n-3}dy;$$

therefore $\dot{\mathrm{J}}$ vanishes, and hence

$$U_{2n-4}=-\frac{1}{2n-3}U_{2n-3};\ \text{similarly}\ U_{2n-3}=-\frac{1}{2n-2}U_{2n-2}\ldots(4,5).$$

We have also $\qquad U_1=-\frac{1}{2}$ ....(6),

as will be found by integrating in

$$U_1=-\tfrac{1}{2}\left\{\frac{\log x}{(1+x)^2}\right\}_0^\infty+\tfrac{1}{2}\int_0^\infty\frac{dx}{x\,(1+x)^2},$$

$$\int_0^\infty\frac{dx}{x\,(1+x)^3}=\int_0^\infty\left\{\frac{dx}{x}-\frac{dx}{(1+x)^3}-\frac{dx}{(1+x)^2}-\frac{dx}{1+x}\right\}.$$

The relations (4), (5), (6) are obviously sufficient for the proof.

II. *Solution by* Prof. TANNER, M.A.; Prof. NASH, M.A.; *and others.*

Separate the proposed general integral into two portions, the former extending between the limits 0, 1, the latter ranging between 1 and ∞. Transform the second portion by writing $\frac{1}{x}$ for x, and after reduction

recombine it with the first part. Thus we have the given integral,

$$I = \int_0^1 \frac{a_0 + a_1(-x) + a_2(-x)^2 + \ldots - a_2(-x)^{n-3} - a_1(-x)^{n-2} - a_0(-x)^{n-1}}{(1+x)^{n+1}} \times (\log x)^{n-} \, dx$$

$$= \int_0^1 \{1 - 2^{n-1}x + 3^{n-1}x^2 - \&c. \text{ ad. inf.}\} (\log x)^{n-2} dx.$$

[This can be verified by comparing coefficients. See also TODHUNTER'S *Hist. of Math. Theory of Probability*, Art. 830.] Observing that

$$\int_0^1 x^r (\log x)^{n-2} dx = (-)^n \frac{(n-2)!}{(r+1)^{n-1}}$$

or

$$\int_0^1 (r+1)^{n-1} x^r (\log x)^{n-2} dx = (-)^n (n-2)!,$$

we have

$$I = (-)^n (n-2)! \, \{1 - 1 + 1 - 1 + \ldots \text{ad inf.}\} = (-)^n \, . \, \tfrac{1}{2} (n-2)!$$

[Prof. TANNER states that the last step is "not unimpeachable." In fact, the PROPOSER devised the integrals in the Question with the express view to test the much-disputed value $\frac{1}{2}$ for the infinite series

$$1 - 1 + 1 - 1 + \ldots ;$$

and it will be seen that, so far as this test goes, the result is justified.]

6189. (By Professor JOHNSON, M.A.)—A pack of cards may be shuffled in the following manner :—Take off with your right hand the top card of the pack (held in your left hand face upward), place the next card on top of it, the next below it, the next on top, and so on till the pack is exhausted. Then, if you take the 13 cards of one suit in order, and shuffle thus 10 times in succession, the cards are restored to their original order. On trial with other numbers of cards, it appeared that, in all cases, as soon as the original top card reappeared on the top of the pack, all the cards are in their original places. Prove this, and find the number of shuffles required for n cards. When an odd number of cards is used, the bottom card does not change its position; the above is therefore the result for 12 cards. But when the number of cards is odd, we obtain a new series of results by placing the second card below the first, the third above, the fourth below, and so on. In particular, 2^n cards or $2^n - 1$ cards require only $n+1$ shuffles. With the full pack theninth shuffle brings all the aces together, all the deuces together, and so on; and three more shuffles (12 in all) restore the original order. A similar arrangement occurs when we use a pack formed of four equal packets. Thus, using 9 cards from each suit, this occurs on the sixth shuffle; three more shuffles in every case restoring the original order. The number of shuffles required are generally very small; a pack of fifty cards, however, requires 50 shuffles.

Solution by Professor LLOYD TANNER, M.A.

We may suppose that there are an even number (n) of cards. Let these be numbered from 1 to n. The effect of a shuffle is to bring 2 before 1, 3 behind 1, 4 before 2, 5 behind 3, and so on. Thus the orders of the cards before and after shuffling are as follows :—

$$1, 2, \ldots, \tfrac{1}{2}n, \tfrac{1}{2}n+1, \ldots n-1, n; \quad n, n-2, \ldots, 4, 2, 1, 3, \ldots n-3, n-1.$$

For a given n the effect of any number of shuffles is most easily found by splitting the substitution just written, into its elementary cyclical substitutions, $(1, n, n-1, n-3, \ldots \tfrac{1}{2}n+1)\,(2, n-2, n-5, \ldots \tfrac{1}{2}n), \ldots$.

For instance, when $n=52$ there are three cycles of the 12th order, and one each of the 6th, 4th, 3rd, 2nd, and 1st orders respectively. The order of the complete substitution is therefore 12, the L.C.M. of the orders of the elementary cycles. To find the effect of r shuffles, we have only to note the rth cards from 1, 2, &c., respectively, each in its proper cycle; these replace 1, 2, &c., after r shuffles.

A somewhat different mode of viewing the question is desirable when n is not known. Let r_m represent the place of a card *before*, r_{m+1} its place *after* a shuffle. Then we find that, according as r_m is odd or even,

$$2r_{m+1} = n + r_m + 1, \ \text{ or } \ n - r_m + 2;$$

that is, $$2\,(2r_{m+1}-1) = 2n+1 \pm (2r_m - 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In this formula there is no ambiguity, for, r_m being given, only one sign will make r_{m+1} integral. The same remark applies to the following equation, which is derived from (1) :—

$$2^m\,(2r_m-1) = (2n+1)\,(2^{m-1} \pm 2^{m-2} \pm \ldots \pm 2 \pm 1) \pm (2r_0-1) \ldots\ldots(2).$$

This equation gives the position (r_m) of a card after m shuffles, the original position being r_0.

Suppose that, after m shuffles, the card in question comes to its original position, so that $r_m = r_0$. Then (2) becomes

$$(2^m \pm 1)\,(2r_0-1) = (2n+1)\,(2^{m-1} \pm 2^{m-2} \pm \ldots \pm 2 \pm 1) = (2n+1)\,.\,\mathrm{M} \ldots(3),$$

where M is an odd number between 0 and 2^m. The smallest value of m which satisfies this equation for all values of r_0 gives the smallest number of shuffles required to reduce the cards to the original order.

Let $r_0 = 1$; then (3) becomes $$2^m \pm 1 = (2n+1)\,.\,\mathrm{M}' \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If $\phi\,(2n+1)$ be the number of numbers less than $2n+1$, and prime to it, one solution of this equation is $m = \phi\,(2n+1)$; but m may be the result of dividing $\phi\,(2n+1)$ by a power of 2. A nearer limit may be obtained when $2n+1$ is a product of factors each of which is a prime or a power of a prime; viz., taking the ϕ of each factor, the L.C.M. of all these ϕ's is either m or the product of m into some power of 2. The proper value of m may be determined by trial, or by a method given in the *Messenger of Mathematics* (Vol. VIII., p. 13).

Supposing m determined for $r_0 = 1$, it can be shown that (3) is satisfied by the same value of m, whatever r_0 may be. For, by means of (4), (3) becomes $$(2r_0-1)\,\mathrm{M}' = \mathrm{M},$$

which gives a value of M within the prescribed limits, as appears on eliminating M′ with the help of (4), and noting that $r_0 \leqq n$. We already know that only one value of M exists. This verifies the statement that, when the top card comes to its original position, the original order is restored.

Examples:—$n = 52$, $\quad 2n+1 = 105 = 3.5.7$;

the value of $\phi(105)$ is 48, while the L.C.M. of $\phi(3)$, $\phi(5)$, $\phi(7)$, is 12. Hence $m = 3$, 6, or 12; and on trial we find that $2^{12}-1$ is, but $2^6 \pm 1$ are not, divisible by 105. Hence $m = 12$.

$n = 2^p$ or $2^p - 1$; (4) becomes $(2^m \pm 1) = (2^{p+1} \pm 1)\,M'$; so that $M' = 1$, $m = p+1$.

$n = 12$, $\qquad \phi(2n+1) = \phi(25) = 20, \quad m = 5$, 10, or 20.

Since $2^{10}+1$, $= 1025$, is a multiple of 25, and $2^5 \pm 1$ are not, $m = 10$.

Of course, for particular values of r_0, (3) may be satisfied by smaller values of m; indeed, this must be the case unless m is a factor of n. For instance, take $n = 52$, $r_0 = 18$, then (3) is satisfied by $m = 1$, becoming $3.35 = 105$. The same equation is satisfied by $m = 2$, $r_0 = 11$ or 32.

To examine the effect of p shuffles upon a pack containing $2^{p-1}.n$ cards (in particular, the effect of 3 shuffles on a pack of $4n$ cards). Making the proper substitutions, and putting $r_0 = 2^{p-1}r + s$, (2) becomes

$$2^p(2r_p - 1) = (2^p n + 1)\,M \pm (2^p r + 2s - 1), \quad \text{where M is odd and} < 2^p;$$

therefore $$2r_p - 1 = n.M \pm r + \frac{M \pm (2s-1)}{2^p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The last term can only be 1 or 0 (since it is integral). When $r \equiv n$ (mod. 2) the last term must be 1 (for the left side is odd). Thus $M = 2^p - 2s + 1$, and the upper sign is to be used. In this case

$$r_p = 2^{p-1}n + 1 + \tfrac{1}{2}(n+r) - s.n.$$

When $r \equiv n+1$ (mod. 2), the last term of (5) must vanish; that is, we must have $M = 2s - 1$, and use the lower sign. Then

$$r_p = s.n - \tfrac{1}{2}(n+r-1).$$

If the pack be arranged in n packets, containing 2^{p-1} aces, deuces, &c. respectively, to each packet belongs one value of r, and to each card in this packet one value of s. The values of r_p just obtained show that after p shuffles the cards of each packet are distributed, each being the nth from its former neighbour. It may also be inferred that, if the final arrangement is to be in sequence, suits attended to, the packets must be arranged thus: at the bottom the aces, then the n's (or "Kings" say), then the deuces, then the "Queens," and so on. Also two consecutive packets must have their suits in opposite orders.

The number of shuffles required to produce this arrangement from the pack arranged in sequences, is $m-p$, m having the signification given in (3).

6169. (By Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR.

Solution by C. J. Monro, M.A.

* Naught, by rights; but signless volume is obviously meant. Thanks, then, to the symmetry of the conditions, we may attend only to the cases in which the volume is positive, and, of course, fix one edge through O in direction, and one face through that edge in aspect. Then the

ranges of P, Q, and R are a semidiameter, a semicircle adjacent, and a hemisphere adjacent to this; and their mean positions are at mutually perpendicular distances $\frac{1}{2}$, $\frac{4}{3\pi}$, $\frac{3}{8}$ of the radius. Since the volume is $\frac{1}{6}$ of the product of the distances of P from O, of Q from OP, and of R from OPQ, the mean positions give a mean tetrahedron, $\frac{1}{24\pi}$ in volume.

6231. (By R. Tucker, M.A.)—Eliminate θ and ϕ from the equations

$$\frac{x}{a}\cos\theta + \frac{y}{b}\cos\phi = \frac{x}{a}\sin\theta + \frac{y}{b}\sin\phi = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \mu^2(\sin 2\theta - \sin 2\phi) \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Solution by C. H. Sampson, M.A.; A. L. Selby, B.A.; *and others.*

From (1), $$\frac{x^2}{a^2} + \frac{2xy}{ab}\cos(\theta-\phi) + \frac{y^2}{b^2} = 2;$$

hence, solving for $\frac{x}{a}$ and $\frac{y}{b}$, we have

$$\frac{x}{a} = \frac{\sin\phi - \cos\phi}{-\sin(\theta-\phi)},\quad \frac{y}{b} = \frac{\sin\theta - \cos\theta}{\sin(\theta-\phi)};$$

therefore $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\sin 2\theta - \sin 2\phi}{\sin^2(\theta-\phi)} = \mu^2(\sin 2\theta - \sin 2\phi);$$

therefore $$\sin^2(\theta-\phi) = \frac{1}{\mu^2} \text{ and } \cos(\theta-\phi) = \frac{\pm(\mu^2-1)^{\frac{1}{2}}}{\mu};$$

therefore $$\frac{x^2}{a^2} \pm \frac{2xy}{ab}\frac{(\mu^2-1)^{\frac{1}{2}}}{\mu} + \frac{y^2}{b^2} = 2.$$

3915. (By J. J. Walker, M.A.)—From points on a diameter of a conic, tangents are drawn and perpendiculars let fall on the chords of contact; prove that the locus of the intersection of these last named lines is a hyperbola having as asymptotes the diameter conjugate to the given one and a perpendicular diameter.

Solution by Professor Wolstenholme, M.A.

The equation of the diameter being $\frac{x}{a\cos\alpha} = \frac{y}{b\sin\alpha}\ (=m)$, the equations of a chord of contact and of the perpendicular thereon are

$$m\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right) = 1,\quad \frac{x - ma\cos\alpha}{\frac{\cos\alpha}{a}} = \frac{y - mb\sin\alpha}{\frac{\sin\alpha}{b}};$$

therefore $$\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}-m\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)}{\frac{x\cos\alpha}{a^3}+\frac{y\sin\alpha}{b^3}}=\frac{bx\sin\alpha-ay\cos\alpha}{\left(\frac{b}{a}-\frac{a}{b}\right)\sin\alpha\cos\alpha},$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{b}{a}-\frac{a}{b}\right)=\left(\frac{x\cos\alpha}{a^3}+\frac{y\sin\alpha}{b^3}\right)\left(\frac{bx}{\cos\alpha}-\frac{ay}{\sin\alpha}\right),$$

or, for the locus, $x^2-y^2+\frac{xy}{ab}(a^2\tan\alpha-b^2\cot\alpha)=a^2-b^2$, which denotes a rectangular hyperbola passing through the four foci of the ellipse, and being in fact the locus of the intersection of tangents equally inclined to a given line, which is in this case the direction of the chord of contact (or the perpendicular); whence we get the following well-known proposition:—If from a point O a perpendicular OP be drawn on its polar with respect to a conic whose foci are S, H, then OP will bisect the angle SPH; or, stated in other words, if a tangent P to an ellipse have the pole O with respect to a confocal, OP will be normal at P.

[Mr. Walker's proof is as follows:—If C be the centre and A the vertex of the given diameter, and if OP meet the conjugate diameter in Q, then $PO.OQ=CA^2\sin OCP\cos OCP=$ constant. In the case of the parabola, the locus is easily seen to be a straight line.]

Note on Question 3915. *By the* Editor.

This Question, it has been pointed out, is a particular case of the following theorem:—From different points of a fixed straight line perpendiculars are drawn upon their polars with respect to a given conic; the locus of the feet of these perpendiculars is the circular cubic which is the locus of the points of contact of tangents drawn from a fixed point (the pole of the fixed straight line) to the conics confocal with the given conic. The proposition stated at the end of the foregoing solution proves this at once; but, without assuming that result, let the equations of the fixed line and fixed conic be $$\frac{px}{a^2}+\frac{qy}{b^2}=1,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let (X, Y) be any point on the fixed line; then the foot of the perpendicular is given by the equations
$$\frac{x-X}{\frac{X}{a^2}}=\frac{y-Y}{\frac{Y}{b^2}}\ (\equiv\lambda);\quad \frac{xX}{a^2}+\frac{yY}{b^2}=1;\quad \text{also } \frac{pX}{a^2}+\frac{qY}{b^2}=1;$$

and, substituting for (X, Y), we get the two equations
$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,\quad \frac{px}{b^2+\lambda}+\frac{qy}{b^2+\lambda}=1;$$

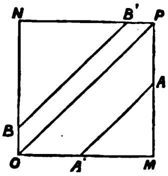
proving that the foot of the perpendicular is a point of contact of a tangent from (p, q) to a conic confocal with the given conic. Eliminating λ, we get the equation of the locus
$$(py-qx)(x^2+y^2-px-qy)=(a^2-b^2)(x-p)(y-q).$$

This is, in general, a circular cubic with a crunode at (p, y). When p and q are infinite, it reduces to the rectangular hyperbola
$$(py-qx)(px+qy)+(a^2-b^2)pq=0.$$

5138. (By the EDITOR).—Two persons make an appointment to meet on a certain day at the Medical Council Office, but without fixing the time further than that it is to be between 2 and 3 o'clock; and each agrees to wait ten minutes for the other. Now, supposing all times between the specified limits for coming to the Office to be equally probable, show, by a general solution, that the odds are 25 to 11 against the meeting taking place.

I. *Solution by* C. J. MONRO, M.A.

To state the question generally, take as unit of time the interval allowed for calling. The chance that A calls between the times x and $x+dx$ is dx, and the chance that B calls between the times y and $y+dy$ is dy. Therefore the chance that they meet is $\iint dx\,dy$ between proper limits. If x and y be counted from the beginning of the interval, we have the limits 0 and 1 for each; and if A. and B. wait for the periods a and b, we have the further limits $x-y=-a$ and b; and here inner limits will supersede outer ones. Proceeding geometrically, for variety's sake, draw a unit square OMPN, and chords AA′, BB′, parallel to OP, such that PA $=a$, OB $=b$. Then the hexagon OA′APB′B will be measured by the above integral, and will therefore, without danger of introducing geometrical conditions foreign to the data, serve to measure the chance required. This, then, is



$$1-\tfrac{1}{2}(1-a)^2-\tfrac{1}{2}(1-b)^2, \quad \text{or} \quad a+b-\tfrac{1}{2}(a^2+b^2),$$

and becomes $2a-a^2$ if $b=a$, or $\frac{11}{36}$ if also $a=\frac{1}{6}$; thus the odds against the meeting are 25 to 11.

II. *Solution by* Prof. SEITZ, M.A.; Prof. NASH, M.A.; *and others.*

Let x and y be the times after 2 o'clock at which the two persons arrive at the office, and $a=\frac{1}{6}$ of an hour the time each agrees to wait for the other. Then, in order that they may meet, the difference between x and y must not exceed a; hence, the chance that they will meet is

$$\int_0^a\int_0^{x+a}dx\,dy+\int_a^{1-a}\int_{x-a}^{x+a}dx\,dy+\int_{1-a}^{1}\int_{x-a}^{1}dx\,dy$$

$$=\int_0^a(x+a)\,dx+\int_a^{1-a}2a\,dx+\int_{1-a}^{1}(1+a-x)\,dx=2a-a^2=\tfrac{11}{36}.$$

III. *Solution by the* Rev. H. G. DAY, M.A.; J. A. KEALY, M.A.; *and others.*

Let the interval for calling be c hours, and let A resolve to wait m hours; then the chance that B comes before A leaves is $\dfrac{m}{c}-\dfrac{1}{2}\dfrac{m^2}{c^2}$.

Similarly, let B resolve to wait n hours; then the chance that A comes before B leaves is $\dfrac{n}{c}-\dfrac{1}{2}\dfrac{n^2}{c^2}$.

Therefore chance of their meeting is $\dfrac{m+n}{c}+\dfrac{1}{2}\dfrac{m^2+n^2}{c^2}$.

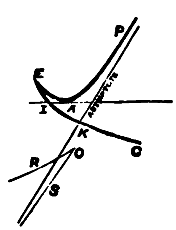
In the given case, we have $m=n=\dfrac{c}{6}$; therefore the chance is $\dfrac{11}{36}$, and the odds are accordingly 25 to 11 against the meeting taking place.

1628. (By Professor SYLVESTER, F.R.S.)—Extending the meaning of the word Bicorn to signify the general curve defined by the equations

$$x = \frac{c^3}{\phi^4+\phi^3},\quad y = \frac{c^2(2\phi+3)}{\phi^3+\phi^2},$$

show that the limiting form of the curve, as c converges towards zero, is a semicubical parabola and its axis extended indefinitely in both directions, forming together a sort of Trident. Explain the mode in which the geometrical passage of the double-horned to the tridentine form takes place, and specify in the Trident the morphological equivalents of the respective four branches of the general curve.

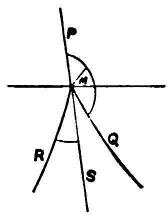
Solution by the PROPOSER.



1. *General Curve.*—The general equation in x, y, c, when c vanishes, becomes

$$x(x^2-ky^3) = 0,$$

k being a known number (in fact $k = \frac{1}{27}$).



2. *Trident.*—Here E, I, K, A, O come together at M, and the Table of Morphological Equivalents is as follows:—

M $\equiv$ E, I, A, K, O,	MS $\equiv$ OS,
PMS $\equiv$ Asymptote,	MQ $\equiv$ EIKQ,
MP $\equiv$ EAP,	MR $\equiv$ OR.

The triangular space EIA vanishes; O moves up to A; EAP becomes MP; OS becomes MS; EIQ becomes MQ; OR becomes MR; and MQ, MR are the two branches of a semicubical parabola, the asymptote becoming PMS.

6229. (By the EDITOR.)—Supposing the value of a rough diamond to vary as the square of its size; find how such a diamond would be reduced in value if it were broken at random into (1) two pieces, (2) three pieces.

I. *Solution by* J. HAMMOND, M.A.; J. A. KEALY, M.A.; *and others.*

Denoting the parts by the letters x, y, z, the conditions are

$$x+y=1,\quad x+y+z=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1, 2).$$

The value of the broken diamond is, in the first case, x^2+y^2, and in the second $x^2+y^2+z^2$; its mean value is therefore the mean value of the square of the radius vector from the origin to all points of (1) or (2), and the mean reduction in value is $2xy$, or $2(yz+zx+xy)$, with the same limits of integration.

In the first case, $2xy = 2x(1-x)$, and the reduction of value is

$$2\int_0^1 x(1-x)\,dx = \frac{1}{3}.$$

In the second case, we have for the reduction $2\int (yz+zx+xy)\frac{dA}{A}$, where A is the area of $x+y+z=1$ intercepted by the planes of x, y, z; or, since $\frac{dA}{A}$ is the same for the area and its projection,

$$2\iint\{xy+(x+y)(1-x-y)\}\frac{dx\,dy}{\frac{1}{2}}$$

(the integration extending over the triangle $x=0,\ y=0,\ x+y=1$)

$$= 4\int_0^1\int_0^{1-x}[x(1-x)+y(1-x)-y^2]\,dx\,dy$$

$$= 4\int_0^1\left[x(1-x)^2+\frac{(1-x)^3}{6}\right]dx = \frac{1}{3}+\frac{1}{6} = \frac{1}{2}.$$

II. *Solution by* C. J. Monro, M.A.

Let a be the size of the whole, and let kx^2 be the value of size x. The probability that, when the size of two pieces together is constant, the size of either is between x and $x+dx$, is doubtless understood to vary as dx.

1. The expectation is obviously $k\int_0^a\{x^2+(a-x)^2\}\frac{dx}{a} = \frac{2}{3}ka^2$, giving a loss of $\frac{1}{3}$. This is equivalent to an actual division into about $\frac{4}{5}$ and $\frac{1}{5}$.

2. This will be greatly simplified by proceeding geometrically. Let $x+y+z=a$ be the equation to a plane: the chance of a particular range of sizes will be measured by the area of the corresponding range of points as a fraction of the area of the triangle enclosing points with positive coordinates. Let dS be an elementary area, and S_1 the whole. Then the expectation is $k\int(x^2+y^2+z^2)\frac{dS}{S_1}$; or, if r (the coordinates being rectangular, is the distance from the perpendicular (of length $\frac{1}{3}a\sqrt{3}$) drawn to the plane from the origin, $k\int\left(\frac{a^2}{3}+r^2\right)\frac{dS}{S_1}$. The second term of the integral will be the square of the radius of gyration of an equilateral triangle of side $\sqrt{2}\,.\,a$, about a normal through its centre, which is $\frac{1}{6}a^2$. So the expectation is $\frac{1}{2}ka^2$, giving a loss of $\frac{1}{2}$. This is equivalent to an actual division into about 13, 5, 2 twentieths.

6013. (By Professor Minchin, M.A.)—If in any point of a body subject to stress the principal stresses consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted. If

the principal stresses are a tension of intensity A and two pressures of intensities B and C (C > B), show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted.

Solution by J. J. WALKER, M.A.

As preliminary to the solution, consider the semi-diameter r to any point (xy) on a hyperbola $a^{-2}x^2 - b^{-2}y^2 = 1$, and p the perpendicular from centre on the tangent at (xy); and let

$$u^2 = r^{-2}(p^{-2} - r^{-2}).$$

Then $\quad a^4b^4u^2 = (b^4x^2 + a^4y^2)(x^2+y^2)^1 - (b^2x^2 - a^2y^2)^2(x^2+y^2)^{-2},$

and $\quad a^{-2}b^{-2} - u^2 = (a^2x^2 + b^2y^2)(b^2x^2 - a^2y^2)\,a^{-2}b^{-2}(x^2+y^2)^{-2},$

an essentially positive quantity, if finite, and which vanishes only when (xy) is the point of contact of an asymptote. The maximum value of u is, therefore, at that limit, when $u^2 = a^{-2}b^{-2}$.

Now, consider the stress-quadric $Ax^2 \pm By^2 - Cz^2 = 1$. The planes across which the force is a simple shear, are those which have the sides of the asymptotic cone $Ax^2 \pm By^2 - Cz^2 = 0$ as normals, and the force across any plane being, in general, proportional to $r^{-1}p^{-1}$ (r being the semi-diameter of the stress-quadric normal to that plane, and p the perpendicular from centre on the tangent plane at the end of r), and in the direction of p, its component parallel to the plane will be $r^{-1}p^{-1}\sin(rp)$, or u, as above. For any one of the simple shears specified, $u^2 = a_1^{-1}b_1^{-1}$, a_1, b_1 being the semi-axes of the hyperbolic section of the stress-quadric made by the plane rp, where r is a side of the asymptotic cone and p the perpendicular at the centre to the tangent plane along that side. Now, lmn being the direction-cosines of the plane of the section,

$$a_1^{-2}b_1^{-2} = (a^2l^2 \pm b^2m^2 - c^2n^2) \div (\pm a^2b^2c^2), \text{ where } a^{-2} = \text{A}\ldots;$$

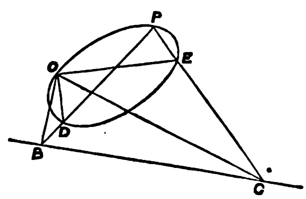
and therefore $\quad a^2b^2c^2(a^{-2}c^{-2} - a_1^{-2}b_1^{-2}) = (b^2 - a^2)\,l^2 + (b^2 + c^2)\,m^2,$

an essentially positive quantity when A > B, or $(a^2+b^2)\,l^2 + (b^2 - c^2)\,n^2$, which is positive for all values of l, n, if C > B. The planes of maximum intensity $(AC)^{\frac{1}{2}}$ of shearing force are therefore those normal to the asymptotic lines $Ax^2 - Cz^2 = 0$.

NOTE ON QUESTION 1409. *By* C. TAYLOR, M.A.

The following theorem is due to the late Professor CLIFFORD:—

Corresponding to every point O on a conic there exists a certain straight line BC without it, such that if chords PD, PE be drawn from any point P on the conic, and be produced to meet BC in B and C, then the angles BOC and DOE are either equal or supplementary. (*Reprint*, Vol. I., pp. 33, 40.)



I propose to show that this affords an illustration of the method of transformation which I have described under the name *Reversion*, in the *Quarterly Journal of Mathematics*, Vol. XIV., pp. 25—39. It is there shown that all angles subtended at the origin O reverse into *equal angles* subtended at the origin ω (p. 33); and hence, that a variable chord of a conic which subtends a right angle at a fixed point O on the curve passes through a fixed point on the normal at O (p. 35), for, if we reverse the conic from O as origin into a *circle*, the variable chord is reversed into a *diameter* of the circle, since it subtends a right angle at the point ω on the circumference.

Hence, conversely, we see that any conic may be reversed into a circle from any point O on the conic as origin *by taking as base-line the polar of the point of concourse of all chords which subtend right angles at* O.

If BC be the line thus related to O in the diagram, the conic reverses into a circle through the reverse origin ω, and (by the nature of reversion) the line at infinity corresponds to BC. Hence, if the points $pde\infty\infty'$ correspond severally to PDEBC; then, since *angles in the same segment of a circle* are equal to one another, therefore

$$\angle d\omega e = dpe = \infty p\infty' = \infty\omega\infty';$$

and, therefore, in the original or obverse figure, the angles DOE and BOC are equal or supplementary, as was to be proved.

Prof. Cayley has pointed out (*Reprint*, Vol. I., p. 40) that the theorem in question may be regarded as a limiting case of Newton's *Descriptio Organica*. For if an angle BOD of given magnitude turn about a fixed point O, and if a straight line PDB, regarded as an angle of *zero* magnitude, turn about a fixed point P; then, if the point of concourse B be made to describe a straight line to BC, the point of concourse D will describe a conic through O and P.

If COE be any second position of the constant angle, then, in the diagram, $\angle \text{DOE} = \text{DOC} + \text{COE} = \text{DOC} + \text{DOB} = \text{BOC}$,
which is Prof. Clifford's theorem; for it may be shown that the line BC has a relation to O which is independent of the position of P on the conic.

To prove the theorem in this second form, by Reversion, let an angle of given magnitude α turn about the origin ω, and a straight line about the origin O; and let this line be always *parallel* to one arm of the constant angle, and meet the other in d, the locus of which is evidently a *circle* through O and ω.

Then, in the obverse figure, we have an angle equal to α turning about O, and a straight line turning about ω (or P), and meeting one arm of the angle on a conic through O and ω, and the other arm on BC the base-line, which is the polar of the point of concourse of all chords of the conic which subtend right angles at O.

The theorem of Quest. 1263 (*Reprint*, Vol. XXXII. p. 29) is deduced from the general case of Quest. 1409 above stated, by making D and E the points at infinity on a rectangular hyperbola.

2668. (By N'Importe.) — A writes to B, requiring an answer within n days. It is known that B will be at the address on some one of these days, any one equally likely. It is a p-days' post between A and B. If one in every q letters is lost in transit, find the chance that A receives an answer in time. $(n > 2p)$.

Solution by J. E. A. STEGGALL, B.A.; C. J. MONRO, M.A.; *and others.*

Call *available* days those on which B would write in time if the post did not fail. It is not quite clear whether the pth is to be an available day or not. Say there are $n-2p+\epsilon$ available days, ϵ being 1 or 0. Then the chance required is the product of the chances that A's letter is not lost $\left(1-\frac{1}{q}\right)$, that B calls on an available day $\left(\frac{n-2p+\epsilon}{n}\right)$, and that B's letter is not lost $\left(1-\frac{1}{q}\right)$; in short $\left(1-\frac{1}{q}\right)^2\frac{n-2p+\epsilon}{n}$. Is it, however, certain B would answer? If w_x is the chance that he is in a good humour on the xth day, the value is

$$\left(1-\frac{1}{q}\right)^2\frac{1}{n}\Sigma_{p+1-\epsilon}^{n-p}w_x.$$

6267. (By C. LEUDESDORF, M.A.)—In a plane triangle R, r, r_1, r_2, r_3 are the radii of the circumscribed, inscribed, and three escribed circles; prove that, if $(r_2+r_3-r_1)(r_3+r_1-r_2)(r_1+r_2-r_3) = -8r_1r_2r_3$, then $r+4R$ will be equal to the perimeter of the triangle.

Solution by the D. EASTWOOD, M.A.; J. OPENSHAW, M.A.; *and others.*

A simple proof may be immediately obtained by aid of the following relations, first proved in the late T. S. DAVIES's *Horæ Geometricæ*, in the *Ladies' Diary* for 1836:—

$$r_2r_3+r_3r_1+r_1r_2 = s^2,\quad r_1+r_2+r_3 = 4R+r = 2\sigma, \text{ suppose};$$

for the given condition thus becomes, at once,

$$(\sigma-r_1)(\sigma-r_2)(\sigma-r_3) = -r_1r_2r_3, \text{ or } \sigma^3-2\sigma.\sigma^2+s^2.\sigma = 0, \text{ whence } \sigma = s.$$

6163. (By Professor MATZ, M.A.)—Prove that the mean distance of all the points of the cardioid-curve from the coordinate origin is $\frac{4}{5}a$.

Solution by G. J. GRIFFITHS, M.A.; Professor NASH, M.A.; *and others.*

The mean distance is $\int_0^s r\,ds \div s$; but, in the cardioid, $r = a(1-\cos\theta)$, and $ds = 2a\sin\frac{1}{2}\theta\,d\theta$; therefore the mean value of r is

$$a\int_0^\pi(1-\cos\theta)\sin\tfrac{1}{2}\theta\,d\theta \div \int_0^\pi\sin\tfrac{1}{2}\theta\,d\theta$$

$$= 2a\int_0^{\frac{1}{2}\pi}\sin^3\theta\,d\theta \div \int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta = 2a\,.\,\tfrac{2}{3} = \tfrac{4}{3}a.$$

6187. (By A. Martin, M.A.)—**Four cards are taken from a full pack; n other cards being drawn, none of them are aces; find the chance that the first four cards drawn were all aces.**

Solution by Prof. Matz, M.A.; W. H. Lowry, M.A.; *and others.*

The final state of the cards is as follows:—There are n exposed cards, none of them aces; and $52-n$ unexposed cards, divided into two lots containing 4 and $52-n-4$ cards respectively. It is required to find the chance that the four aces are all together in the small lot; this chance is clearly $4! \div \{(52-n)(51-n)(50-n)(49-n)\}$, the reciprocal of the number of ways in which 4 cards can be drawn from a pack of $52-n$ cards.

5329. (By Professor Townsend, F.R.S.)—**A rigid body, in unconstrained equilibrium in free space, being supposed set in motion by a single impulsive force applied at a definite point of its mass; if the initial motion be a pure rotation, show that—**

(*a*) **The several possible lines of impulsive action through the point generate a quadric cone, containing the three perpendiculars from the point on the three central principal planes, and the connector of the point with the centre of inertia of the body.**

(*b*) The several corresponding axes of initial rotation in the body envelope a quadric cone, touching the three central principal planes, and the diametral plane conjugate to the direction of the point with respect to the central ellipsoid of the body.

Solution by the Proposer.

Denoting by M the mass of the body, by abc its three principal arms of central inertia, by xyz the coordinates of the point with respect to its three central principal planes, by F the impulsive force, by $\alpha\beta\gamma$ the direction angles of its action at xyz, by uvw and pqr the components, linear and angular, of the initial velocities produced by its action at and round the centre of inertia of the body, and by $\xi\eta\zeta$ the coordinates of any point on the initial axis of twist resulting from the composition of both motions; then, since, on the hypothesis of initial pure rotation,

$$u+q\zeta-r\eta=0,\quad v+r\xi-p\zeta=0,\quad w+p\eta-q\xi=0;$$

therefore, substituting for uvw and pqr their known dynamical values, we have the three equations

$$\cos\alpha+b^{-}\ (z\cos\alpha-x\cos\gamma)\,\zeta-c^{-2}\,(x\cos\beta-y\cos\alpha)\,\eta=0,$$
$$\cos\beta+c^{-2}\,(x\cos\beta-y\cos\alpha)\,\xi-a^{-2}\,(y\cos\gamma-z\cos\beta)\,\zeta=0,$$
$$\cos\gamma+a^{-2}\,(y\cos\gamma-z\cos\beta)\,\eta-b^{-2}\,(z\cos\alpha-x\cos\gamma)\,\xi=0;$$

and from these, by elimination, first of $\xi\eta\zeta$, and then of pqr, the relations

$$p\cos\alpha+q\cos\beta+r\cos\gamma=0,\quad\text{and}\quad \xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma=0,$$

which prove both parts of the proposed question.

For, substituting in the first for pqr their dynamical values, we get between $\alpha\beta\gamma$ the relation

$$a^2(b^2-c^2)x \,.\, \cos\beta\cos\gamma + b^2(c^2-a^2)y \,.\, \cos\gamma\cos\alpha + c^2(a^2-b^2)z \,.\, \cos\alpha\cos\beta = 0,$$

which is, consequently, the equation of the cone generated by the several lines of impulsive action of the force; and forming, subject to this latter relation, the equation of the envelope of the plane represented by the second for any values of $\alpha\beta\gamma$, we get between $\xi\eta\zeta$ the relation

$$a(b^2-c^2)^{\frac{1}{2}}x^{\frac{1}{2}} \,.\, \xi^{\frac{1}{2}} + b(c^2-a^2)^{\frac{1}{2}}y^{\frac{1}{2}} \,.\, \eta^{\frac{1}{2}} + c(a^2-b^2)^{\frac{1}{2}}z^{\frac{1}{2}} \,.\, \zeta^{\frac{1}{2}} = 0;$$

which is, accordingly, the equation of the cone enveloped by the several axes of initial pure rotation of the body; and therefore, &c., as regards both parts of the question.

Eliminating $\alpha\beta\gamma$ from the three equations of the original groups, we get, for the ruled surface generated by the several axes of rotation, the equation

$$\begin{vmatrix} +(1+b^{-2}\,.\,z\zeta+c^{-2}\,.\,y\eta), & -c^{-2}\,.\,x\eta, & -b^{-2}\,.\,x\zeta \\ -c^{-2}\,.\,y\xi, & +(1+c^{-2}\,.\,x\xi+a^{-2}\,.\,z\zeta), & -a^{-2}\,.\,y\zeta \\ -b^{-2}\,.\,z\xi, & -a^{-2}\,.\,z\eta, & +(1+a^{-2}\,.\,y\eta+b^{-2}\,.\,x\xi) \end{vmatrix} = 0,$$

which surface is, consequently, a cubic scroll, enveloped from the centre of inertia of the body by the quadric cone, the equation of which has been just given.

Note.—If, in the several preceding relations, $\alpha\beta\gamma$ were supposed constant, and xyz instead variable, it would manifestly follow at once from these that, under the same initial circumstances of the body, the several possible lines of action of the single impulsive force, and the several corresponding axes of the initial pure rotation, would generate planes at right angles to each other, whose equations, referred to the three central principal planes of the body, would be $a^2(b^2-c^2)\cos\beta\cos\gamma\,.\,x + \&c. = 0$, and $\cos\alpha\,.\,\xi + \&c. = 0$; properties which were already proved on other principles in No. 4370, page 107, Vol. XXI., of the *Educational Times Reprint* for the year 1874.

6177. (By B. Williamson, M.A.)—Suppose F, F_1, F_2 to be the three foci of a Cartesian oval, F_2 being the exterior focus, and let $FF_1 = c_2$, $FF_2 = c_1$, $F_1F_2 = c$; then, if $mr + lr_1 = nc_2$ be the equation of the inner oval referred to F and F_1, show (1) that its equation referred to F and F_2 is $nr + lr_2 = mc_1$; and, referred to F_1 and F_2, is $mr_2 - nr_1 = lc$. Also prove (2) that the corresponding equations of the outer oval are obtained by changing l into $-l$ in the preceding.

Solution by J. J. Walker, M.A.

This question has been substantially answered by the results of my solution of Prof. Sylvester's Question 4922 (*Reprint*, Vol. XXV., p. 69), wherein, forming the equivalent polar equations with F as pole, and FF_1F_2 as prime radius, it is shown that

$$mr + r_1 = s, \text{ and } \mu r + r_2 = \sigma \text{........................(1, 2)},$$

will represent the same curve, if

$$s = \mu c_2(c),\ \ \sigma = mc_1(\gamma),\ \ (m^2-1)\,c_1 = (\mu^2-1)\,c_2,\ \text{ or }\ m^2c_1 - \mu^2c_2 = c_1 - c_2 = c;$$

so that, substituting $\frac{m'}{l}$ for m, and $\frac{n}{l}$ for μ, (1) and (2) become

$$mr + lr_1 = nc_2, \quad nr + lr_2 = mc_1 \text{ respectively.}$$

By elimination of r between these, in virtue of the relation between cc_1c_2, the third form $mr_2 - nr_1 = lc$ is obtained.

It further appears, that, for a given value of θ, there are two values of r (say r, r'), such that $rr' = c_1c_2$ $(c\gamma)$, one corresponding to $mr + lr_1 = nc_2$, the other to $mr' - lr'_1 = nc_2$; since the elimination of r_1 from the former, and that of r'_1 from the latter, lead to one and the same univectorial polar equation. By eliminating nc_2 between these, $m(r'-r) = l(r'_1+r)$, which shows that the oval corresponding to the negative value of l is exterior to the other. [See also WILLIAMSON's *Diff. Calc.*, 3rd ed., pp. 411—416.]

5432. (By Professor BALL, LL.D.)—If from any point perpendiculars are drawn to the generators of the surface $z(x^2+y^2) - 2mxy = 0$; show that the feet of the perpendiculars lie upon a plane ellipse.

Solution by J. J. WALKER, M.A.

The surface is defined as generated by the line

$$\lambda x - y = 0, \quad (1+\lambda^2)z = 2\lambda m,$$

λ being a variable parameter. The plane through $(x'y'z')$, perpendicular to each of the above planes, is $x + \lambda y = x' + \lambda y'$. It is easily found that the point common to these three planes is

$$(1+\lambda)^2 x = x' + \lambda y', \quad (1+\lambda^2) y = \lambda(x' + \lambda y'), \quad (1+\lambda)^2 z = 2\lambda m;$$

and that these coordinates satisfy the equations

$$x^2 + y^2 = x'x + y'y, \quad 2my'x + 2mx'y - (x'^2 + y'^2)z = 2mx'y',$$

a circular cylinder and plane respectively.

[Another solution is given on p. 96 of Vol. XXX. of the *Reprint.*]

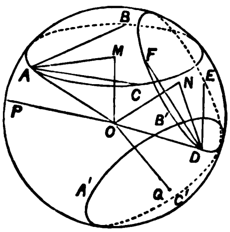
6264. (By R. E. RILEY, B.A.)—If a circle be drawn with its centre at the vertex of the parabola $(x-y)^2 = (32)^{\frac{1}{2}} a(x+y)$, and with a diameter equal to the latus-rectum, prove that the poles with respect to the parabola, of tangents to the circle, lie on the rectangular hyperbola $xy = 2a^2$.

Solution by Prof. SCOTT, M.A.; J. E. A. STEGGALL, B.A.; *and others.*

Turning the axes through 45°, we get, for the parabola, $y^2 = 4ax$; and for the circle, $x^2 + y^2 = 4a^2$; and the polar of (hk) is $yk - 2ak = 2ah$, which touches the circle if $k^2 + 4a^2 = h^2$; changing back our axes, we find that (kh) lies on $xy = 2a^2$.

5507. (By Elizabeth Blackwood.)—If A, B, C, D, E, F are random points on the surface of a sphere, find the probability that the plane through A, B, C will intersect the plane through D, E, F inside the sphere.

Solution by Prof. Seitz, M.A.; Prof. Matz, M.A.; *and others.*



Let M and N be the centres of the sections made by the planes through A, B, C and D, E, F; O the centre of the sphere; OP a radius such that AB is parallel to the plane MOP; OQ a radius such that DE is parallel to the plane NOQ; and A′B′C′ a section of the sphere perpendicular to OQ, and equal to the section ABC. Now, if we give the six points all possible positions, it is evident that the whole number of intersections of the circles ABC and DEF is equal to the whole number of the intersections of the circles A′B′C′ and DEF; hence we will consider those positions of the points in which the circles A′B′C′ and DEF intersect.

Let $OA = r$, $\angle AOM = \theta$, $\angle DON = \phi$, $\angle BAM = \mu$, $\angle CAM = \nu$, $\angle MOP = \lambda$; the angle the plane POM makes with a fixed plane through $PO = \rho$, $\angle EDN = \psi$, $\angle FDN = \omega$, $\angle NOQ = \tau$, and diedral $MOQN = \chi$. Then an element of surface at A is $2\pi r^2 \sin\theta\, d\theta$, at B it is $4r^2 \sin\theta \cos\mu\, d\mu\, d\lambda$, at C it is $4r^2 \sin\theta \sin\lambda \cos\nu \sin(\mu+\nu)\, d\nu\, d\rho$, at D it is $2\pi r^2 \sin\phi\, d\phi$, at E it is $4r^2 \sin\phi \cos\psi\, d\psi\, d\tau$, and at F it is $4r^2 \sin\phi \sin\tau \cos\omega \sin(\psi+\omega)\, d\omega\, d\chi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; of μ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ν, $-\mu$ and $\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of ρ, 0 and 2π; of ψ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ω, $-\psi$ and $\frac{1}{2}\pi$, and doubled; of τ, $\pm(\theta-\phi)$ and $\theta+\phi$ (the double sign being taken + when $\theta>\phi$, and − when $\theta<\frac{1}{2}$); of χ, 0 and 2π.

Hence, since the whole number of ways the six points can be taken, is $(4\pi r^2)^6$, the required chance is

$$p = \frac{4}{(4\pi r^2)^6}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi}\int_{\pm(\theta+\phi)}^{\theta+\phi}\int_0^{2\pi} 32\pi r^6 \sin^3\theta\, d\theta \,.\, 32\pi r^6$$
$$\times \sin^3\phi\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi$$
$$\times \cos\omega \sin(\psi+\omega)\, d\omega \sin\tau\, d\tau\, d\chi$$

$$= \frac{2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} \sin^3\theta \sin^3\phi\, d\theta\, d\phi \cos\mu\, d\mu$$
$$\times \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi \cos\omega \sin(\psi+\omega)\, d\omega \sin\tau\, d\tau$$

$$= \frac{4}{\mu^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu$$
$$\times \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi \cos\omega \sin(\psi+\omega)\, d\omega$$

$$= \frac{1}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$
$$\times \sin\lambda\, d\lambda\, d\rho\, (\pi + 2\psi + 2\cot\psi) \sin\psi \cos\psi\, d\psi$$

$$= \frac{3}{2\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$
$$\times \sin\lambda\, d\lambda\, d\rho$$

$$= \frac{3}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \cos\mu \, d\mu \cos\nu \sin(\mu+\nu)\, d\nu \times \sin\lambda \, d\lambda$$

$$= \frac{6}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \cos\mu \, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$

$$= \frac{3}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \, (\pi + 2\mu + 2\cot\mu) \sin\mu \cos\mu \, d\mu$$

$$= \frac{9}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi = \frac{27\pi}{64}\int_0^{\frac{1}{2}\pi} \sin^4\theta \, d\theta = \frac{81\pi^2}{1024}.$$

6192. (By Professor Wolstenholme, M.A.)—Prove that any uniform lamina of mass nM is kinetically equivalent to a system of n equal particles, each of mass M, placed on the ellipse $Ax^2 + By^2 = 2AB$, at points whose successive eccentric angles differ by $\frac{2\pi}{n}$, the axes being the principal axes at the centroid, and nMA, nMB the principal moments of inertia.

Solution by G. J. Griffiths, M.A.; W. J. Constable, M.A.; *and others.*

Let the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be described on a lamina with its centre coinciding with the centroid, and its axis with the principal axes, of the lamina. Let n particles each of mass m be placed on the ellipse, at points whose eccentric angles differ by $\frac{2\pi}{n}$, θ being the eccentric angle of the first; then the moment of inertia of the particles about the major axis is

$$Mb^2 . \Sigma_{r=0}^{r=n-1} \sin^2\left(\theta + \frac{2r\pi}{n}\right) = Mb^2 . \tfrac{1}{2}n.$$

So the moment of inertia about the minor axis $= \frac{1}{2}nMa^2$. The product of inertia $\equiv \frac{1}{2}Mab\Sigma_{r=0}^{r=n-1} \sin 2\left(\theta + \frac{2r\pi}{n}\right)$ vanishes. So that the axes of the ellipse are principal axes for the system of particles as well as for the lamina. Again, since $\Sigma_{r=0}^{r=n-1}\cos\left(\theta + \frac{2r\pi}{n}\right)$ and $\Sigma_{r=0}^{r=n-1}\sin\left(\theta + \frac{2r\pi}{n}\right)$ both vanish, the C. G. of the particles coincides with that of the lamina. Let nM be the mass of the lamina, and nMA, nMB the principal moments of inertia; then the system of particles is kinetically equivalent to the lamina, provided $a^2 = 2B$, $b^2 = 2A$, that is, if the points lie on the ellipse

$$Ax^2 + By^2 = 2AB.$$

6234. (By R. Rawson.)—When the roots of $x^3+nx^2+ax+b=0$ are all real; show, without the aid of Cardan's formula, that

$$4bn^3-a^2n^2-18abn+4a^3+27b^2<0,$$

and that one of the roots is less than $\frac{1}{3}\left\{2(n^2-3a)^{\frac12}-n\right\}$.

Solution by the Proposer; A. Anderson; *and others.*

If r be a real root of the given cubic (1), the remaining two roots will be determined from the quadratic

$$x^2+(r+n)x+r^2+nr+a=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

hence, the roots of (1) are all real when the roots of (2) are real, that is, when $\quad(r+q)^2>4(r^2+nr+a)$, or $\frac{1}{3}\left\{2(n^2-3a)^{\frac12}-n\right\}>r$.
Since r is a root of (1), we have

$$r^2+nr+a=-\frac{b}{r},\text{ therefore } r=-\frac{n}{2}\pm\left(\frac{n^2}{4}-a-\frac{b}{r}\right)^{\frac12};$$

thus $\quad \frac{1}{3}\left\{2(n^2-3a)^{\frac12}-n\right\}>-\frac{n}{2}\pm\left(\frac{n^2}{4}-a-\frac{b}{r}\right)^{\frac12};$

hence $\quad \left\{\frac{2}{3}(n^2-3a)^{\frac12}+\frac{n^2}{6}\right\}^2+a-\frac{n^2}{4}>-\frac{b}{r};$

therefore $\quad \left\{\left[\frac{1}{3}(n^2-3a)^{\frac12}+\frac{n^2}{6}\right]^2+a-\frac{n^2}{4}\right\}\left[\frac{2(n^2-3a)^{\frac12}-n}{3}\right]>-b;$

which reduces to $\quad 4bn^3-a^2n^2-18abn+4a^3+27b^2<0.$

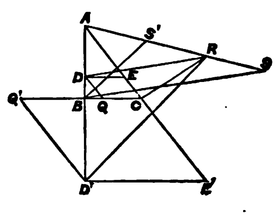
If $n=0$, the roots of the cubic $x^3+ax+b=0$ are all real when

$$4a^3+27b^2<0.$$

[Mr. Rawson believes the above method of determining the condition for the real roots of the cubic to be "preferable to the usual method of infinite series in extracting the cube root of a binomial ($A+iB$) as used by Mr. Todhunter in his *Theory of Equations*."]

5706. (By the Editor.)—Parallel to the base BC of a triangle ABC draw a straight line DE, cutting the sides AB, AC in D, E, such that the squares on BD and CE shall be together equal to the square on DE.

Solution by the Proposer.



Draw CR perpendicular to AC and equal to AB; on AR take RS = RS' = BC; draw RD, RD' parallel, respectively, to SB, S'B, and meeting AB in D, D'; then the points D, D' will determine the position of two parallels DE, D'E' to the base BC, both of which will satisfy the conditions of the problem.

For, draw DQ, D'Q' parallel to AC;

then we have $\quad CQ^2 : QB^2 = AD^2 : DB^2 = AR^2 : RS^2$
$\qquad\qquad\qquad\qquad = AB^2 + AC^2 : BC^2 = BD^2 + DQ^2 : QB^2$;

therefore $\quad BD^2 + DQ^2 = CQ^2$, or $BD^2 + CE^2 = DE^2$.

The same proof obviously applies to $D'E'$.

If A is a right angle, so that $AB^2 + AC^2 = BC^2$, the points Q, Q′ are the internal and external bisectors of BC, and D, D′ the internal and external bisectors of AB.

6210. (By J. L. McKenzie, M.A.)—Prove that the conics of 5-point contact at any three collinear points on a cubic, meet the curve again in three collinear points.

Solution by L. H. Rosenthal, B.A.; A. L. Selby, M.A.; *and others.*

If of the $3(m+n)$ points of intersection of a curve of the $(m+n)$th order with a cubic, $3m$ lie on a curve of the mth order, the remaining $3n$ lie on a curve of the nth order. Hence, regarding the three conics as a sextic, 15 of the points of intersection with the cubic lie on a curve of the 5th order, viz., on 5 coincident right lines; therefore the remaining 3 intersections are collinear.

6190. (By Professor. Townsend, F.R.S.)—A uniform flexible cord, passing without friction through two fixed eyelets in its space, being supposed to hang between them in an arc of an equilateral hyperbola, and beyond them in the two intercepting radii of the curve, its two extremities just reaching the centre; show that the supposed position will be one of free equilibrium under the action of a uniform attractive force emanating from the centre of the curve.

Solution by Prof. Moncourt; J. Hammond, M.A.; *and others.*

The necessary and sufficient conditions for the equilibrium of a string under the action of any central force are

$$(1)\ Tp = \text{const. and } dT = P\,ds\,.\cos\phi, \text{ or } (2)\ dT = P\,dr;$$

of which (1) expresses the fact that the resultant of the tensions at the extremities of any portion of the string passes through the centre of force; while (2) is the condition that the tangential component of the force $P\,ds$, which acts on the element ds of the string, is balanced by the difference of the tensions at its extremities. Now, for the equilibrium of the straight portions of the string, we have, since P is constant $= \mu$, and $T = 0$ when $r = 0$, the free extremity being at the centre, $T = \mu r$; and, since there is no friction, this condition also holds along the curve. Substituting in (1), we obtain (3) $pr =$ const. Now, in the equilateral hyperbola $x^2 - y^2 = a^2$

we have $pr = a^2$, and it is easy to show that, r being variable, the equilateral hyperbola is the only possible form of free equilibrium which satisfies equation (3).

The case of r invariable is a singular solution of the equation formed by eliminating T between (1) and (2), and the circle is a possible form of equilibrium for all laws of force, provided only that the force be repulsive. In fact, since $dr = 0$, (2) gives $T =$ const. for all values of P, and therefore, by (1), $p =$ const., and the constants must be determined so that $p = r$.

[Prof. TOWNSEND remarks that, "in the equilateral hyperbola, regarded as the free catenary of a uniform flexible cord under the action of an attractive force emanating from the centre of the curve, the maintaining force being independent of the distance, and the accompanying tension being that to the centre under its action, throughout the entire extent of any arc of the curve, therefore, &c."]

5968. (By Professor BENJAMIN PEIRCE, LL.D., F.R.S.)—If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction.

Solution by C. J. MONRO, M.A.

Let P, m, ρ, P$'$, m', ρ' be the bodies, their masses, their vectors from C, the centre. Then k, K, K$'$ being scalars, and K and K$'$ independent of ρ and ρ', the forces on P and P$'$ may be written

$$k(\rho'-\rho)-\mathrm{K}\rho,\quad k(\rho-\rho')-\mathrm{K}'\rho';$$

and the acceleration of P$'$ relative to that of P will be

$$k\left(\frac{1}{m}+\frac{1}{m'}\right)(\rho-\rho')+\frac{\mathrm{K}}{m}\rho-\frac{\mathrm{K}'}{m'}\rho',$$

or

$$\left\{k\left(\frac{1}{m}+\frac{1}{m'}\right)+\frac{\mathrm{K}}{m}\right\}\rho-\left\{k\left(\frac{1}{m}+\frac{1}{m'}\right)+\frac{\mathrm{K}'}{m'}\right\}\rho'.$$

If the force towards C is independent of mass, $\mathrm{K}'=\mathrm{K}$, and the coefficients of the vectors may be in any ratio; therefore, with deference to the distinguished Proposer, I do not see that the relative orbit is under any restriction but that of having CPP$'$ for instantaneous plane, which it would with any laws of attraction. But if the *acceleration* towards C is independent of mass, or the force *proportional* to mass, $\dfrac{\mathrm{K}'}{m'}=\dfrac{\mathrm{K}}{m}$, and the acceleration, being $\rho-\rho'$ multiplied by a scalar, will be towards P, and the apparent motion will be as if there were no third centre of attraction. The result would be the same if, instead of the "centre," we had a "body" attracting with a force proportional to its mass. But a particular case makes the literal interpretation improbable. If the *force* towards C is independent of mass, we may have P as nearly as we please constantly at rest, by making m great enough, and putting P at rest for once; but thus, as nearly as we please, P$'$ will simply be a body moving under forces towards two fixed points, with no other limitation than that the force towards one of the points is as the distance. I suppose the proposition will not be true, in general, of this case.

5762. (By Professor SYLVESTER, F.R.S.)—If $f(x, y)$ is a quantic of n dimensions in x, y, prove that $(n-1)\left(\frac{df}{dx}\right)^2 - nf\frac{d^2f}{dx^2}$ is a negative numerical multiplier of y^2 into the Hessian of f.

Solution by W. J. CURRAN SHARP, M.A.

$$H = \begin{vmatrix} u_{11}, & u_{12} \\ u_{12}, & u_{22} \end{vmatrix} = \frac{1}{y}\begin{vmatrix} u_{11}, & yu_{12} \\ u_{12}, & yu_{22} \end{vmatrix} = \frac{n-1}{y}\begin{vmatrix} u_{11}, & u_1 \\ u_{12}, & u_2 \end{vmatrix}$$

[since $xu_{11}+yu_{12} = (n-1)u_1$, and $xu_{12}+yu_{22} = (n-1)u_2$]

$$= \frac{n-1}{y^2}\begin{vmatrix} u_{11}, & u_1 \\ (n-1)u_1, & nu \end{vmatrix} = \frac{n-1}{y^2}\left\{nuu_{11}-(n-1)u_1^2\right\},$$ the form required.

A similar reduction of the Hessian of a ternary n-ic leads to the result

$$H = \frac{n-1}{z^2}\begin{vmatrix} u_{11}, & u_{12}, & u_1 \\ u_{12}, & u_{22}, & u_2 \\ (n-1)u_1, & (n-1)u_2, & nu \end{vmatrix}$$

which, being independent of any differentiation with respect to z, is peculiarly applicable when Cartesian coordinates are used.

[The Hessian of the quaternary n-ic admits of a similar reduction, which is applicable when the coordinates are Cartesian, the result being

$$H = \frac{n-1}{w^2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_1 \\ u_{12}, & u_{22}, & u_{23}, & u_2 \\ u_{13}, & u_{23}, & u_{33}, & u_3 \\ (n-1)u_1, & (n-1)u_2, & (n-1)u_3, & nu \end{vmatrix}$$

The foregoing solution is more elegant than that given on p. 34 of this volume.]

5753. (By W. R. ROBERTS, M.A.)—Show that the equation of the three tangents at the points of intersection of the straight line and cubic

$$L \equiv \lambda x + \mu y + \nu z, \quad U \equiv x^3 + y^3 + z^3 + 6mxyz,$$

is
$$\Sigma U - (1+8m^3)L^2M = 0,$$

where Σ is the condition that L should touch U, and

$$M \equiv [\lambda^4 - 2\lambda(\mu^3+\nu^3) - 6m\mu^2\nu^2]x + [\mu^4 - 2\mu(\lambda^3+\nu^3) - 6m\lambda^2\nu^2]y + [\nu^4 - 2\nu(\lambda^3+\mu^3) - 6m\lambda^2\mu^2]z.$$

[For the value of Σ, see SALMON's *Higher Plane Curves*, p. 74, 2nd Ed.]

Solution by W. J. CURRAN SHARP, M.A.

If the third evectant of the covariant AL'^3 + &c. (SALMON's *Higher Plane Curves*, p. 195) be taken, it will be found to be

$$6L^3\left\{\Sigma U - (1+8m^3)L^2M - 8PLF\right\},$$

where F is the polar conic of L—so that M is a covariant of the cubic and L—and being of the 4th degree in $\alpha\beta\gamma$, it must be the satellite of L.

Now, $AU + BL^2M = 0$ represents a cubic touching U at the points where it meets L; and if L be a tangent, *i.e.*, if λ, μ, ν satisfy $\Sigma = 0$, the system reduces to L^2M, and therefore $A = \Sigma$, and if the tangents at the points where the curve is cut by z, B will be found $= -(1+8m^3)$, and $\Sigma U - (1+8m^3) L^2M = 0$ is the equation to the system of tangents. If L be infinity, this is the equation to the asymptotes, and M is the satellite to infinity. By this means the reality of the asymptotes may be determined

[The result may also be obtained by forming the equation

$$(L_1 x + M_1 y + N_1 z)(L_2 x + M_2 y + N_2 z)(L_3 x + M_3 y + N_3 z) = 0,$$

by the help of the binary cubic in SALMON's *Higher Plane Curves*, p. 74.]

NOTE TO THE SOLUTION OF QUEST. 5753. *By* W. J. C. SHARP, M.A.

This interesting question suggests the problem of forming the equation to the n tangents at the points where an n-ic is met by the line $\alpha x + \beta y + \gamma z$. This equation will evidently be the eliminant of the three equations

$$U = 0, \quad L\xi + M\eta + N\zeta = 0, \quad \alpha x + \beta y + \gamma z = 0,$$

and will therefore be of the n^{th} order in $(\xi\eta\zeta)$ the current coordinates, of the $n(n-1)^{\text{th}}$ in the quantities $\alpha\beta\gamma$, and of $(2n-1)^{\text{th}}$ in the coefficients of U. It will, in fact, be of the form $\Sigma . U + (\alpha x + \beta y + \gamma z)^2 S = 0$, where $\Sigma = 0$ is the tangential equation to the curve, and S the satellite of $\alpha x + \beta y + \gamma z$. For ΣU will be of the proper order in all the quantities, and so will the other term, if S is of the $(2n-1)^{\text{th}}$ order in the coefficients, of $(n+1)(n-2)^{\text{th}}$ order in $\alpha\beta\gamma$, and of the $(n-2)^{\text{th}}$ order in the coordinates, as it should be; while the equation represents an n-ic which touches U at its points of intersection with $\alpha x + \beta y + \gamma z$, and contains that line as a double factor if it be a tangent, and not otherwise. The result of the elimination also gives the equation to the satellite.

6247. (By Professor WOLSTENHOLME, M.A.)—A point O is taken within a triangle ABC, and circles are drawn with centres A, B, C and radii OA sin BOC, OB sin COA, and OC sin AOB respectively; prove that the internal common tangents to these circles, taken two and two, will pass three of them through O, and the other three through another point O′, such that O, O′ are foci of an inscribed ellipse. [If O be taken outside, one pair of internal and two pair of external tangents must be taken.]

I. *Solution by* Professor TOWNSEND, F.R.S.

Denoting by a, b, c the three sides of the triangle ABC, by l, m, n the three lines OA, OB, OC, and by α, β, γ the three angles BOC, COA, AOB respectively; then, the lengths of the three tangents from O to the three

circles being $l \cos \alpha$, $m \cos \beta$, $n \cos \gamma$, and the squares of their three differences in pairs, viz., $(m \cos \beta - n \cos \gamma)^2$, &c., being consequently equal to the squares of the lengths of the internal common tangents to the corresponding pairs of circles, viz., $a^2 - (m \sin \beta + n \sin \gamma)^2$, &c., or

$$m^2 + n^2 - 2mn \cos \alpha - (m \sin \beta + n \sin \gamma)^2, \text{ \&c.},$$

or
$$m^2 + n^2 - 2mn \cos (\beta + \gamma) - (m \sin \beta + n \sin \gamma)^2, \text{ \&c.};$$

therefore &c., as regards the first, and by consequence as regards the second, part of the property in question.

Three circles of any radii being supposed to have their centres at the three vertices of a triangle of any form, it may be easily shown that their three pairs of internal common tangents, or, more generally, any three of their six pairs of common tangents whose intersections are not collinear, are three pairs of tangents to a common conic confocal with that touching the three sides of the triangle at the three corresponding centres of similitude of the three pairs of circles; so that when, as in the question, three of the six tangents pass through a common point, the remaining three also pass through a common point, and the two common points are the two foci of the conic which touches, as aforesaid, the three sides of the triangle, at the three corresponding centres of similitude on its sides.

If a, b, c be the three sides of the triangle, and p, q, r the radii of three opposite circles, it may be easily shown that the equation

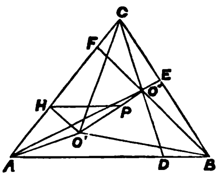
$$[a^2 - (q+r)^2]^{\frac{1}{2}} + [b^2 - (r+p)^2]^{\frac{1}{2}} + [c^2 - (p+q)^2]^{\frac{1}{2}} = 0$$

expresses the condition that the conic touched by the six double tangents should break up into a point-pair; for, if u, v, w be the lengths of the three tangents from either point O of the pair, since then, evidently,

$$u - w = [a^2 - (q+r)^2]^{\frac{1}{2}}, \quad w - u = [b^2 - (r+p)^2]^{\frac{1}{2}}, \quad u - v = [c^2 - (p+q)^2]^{\frac{1}{2}};$$

therefore, at once, the relation in question.

II. *Solution by* R. Rawson; Prof. Hall, M.A.; *and others.*



Let CD, AE, BF pass through O; then

sin BCO . sin CAO . sin ABO
= sin ACO . sin BAO . sin CBO ......(1).

Take

$\angle$ ACO′ = BCO, ABO′ = CBO, BAO′ = CAO, ...(2);

then, by (1), CO′, BO′, AO′ pass through O′.

[The above may be seen neatly demonstrated on p. 260 of Vol. II. of Booth's *New Geometrical Methods.*]

If O, O′ be regarded as the foci of an ellipse, then AC, BC will touch it, because $\angle$ ACO′ = BCO; and BC, AB will touch it, because $\angle$ CBO = ABO′; hence the ellipse whose foci are O, O′, and semi-axis transverse PH (P being the middle of OO′, and O′H perpendicular to AC), is inscribed in the triangle ACB.

Again, $\quad \text{AO} = \text{BO}\dfrac{\sin \text{ABO}}{\sin \text{BAO}}; \quad$ and $\quad \sin \text{BOC} = \dfrac{\text{BC}}{\text{BO}} . \sin \text{BCO};$

therefore $\quad \text{AO} . \sin \text{BOC} = \dfrac{\text{BC} . \sin \text{ABO} . \sin \text{BCO}}{\sin \text{BAO}}$ ....(3),

and $\quad \text{AO}' \sin \text{BO}'\text{C} = \dfrac{\text{BC} . \sin \text{ABO}' . \sin \text{BCO}'}{\sin \text{BAO}'}$ ....(4).

From (1) and (2), AO . sin BOC = AO′ . sin BO′C ....(5);
similarly, BO . sin AOC = BO′ . sin AO′C.........................(6),
CO . sin AOB = CO′ . sin AO′B.........................(7).

Hence there are two points O, O′ within the triangle ABC by which the same circles can be drawn as described by the question.

From O draw a tangent to the circle whose centre is A, and radius = AQ = AO . sin BOC = AO . sin BOD;

therefore $\sin AOQ = \dfrac{AQ}{AO} = \sin BOD$, therefore $\angle AOQ = BOD$.

From O draw a tangent to the circle whose centre is B, and radius = BR = BO sin AOC = BO . sin AOD;

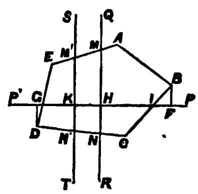
therefore $\sin BOR = \dfrac{BR}{BO} = \sin AOD$, therefore $\angle BOR = AOD$.

Hence OQ coincides with OR, which is a common tangent to the circles at A and B. Therefore the common internal tangent to the circles at A, B, drawn as in the question, will pass through O. Similarly, the common tangents to the circles at B, C, and A, C will pass through O. The same inferences apply to the point O′.

Similar principles and investigation as the above will apply equally well when the point O is outside the triangle ACB. If O is outside, then O′ is also outside; and, in this case, the points O, O′ may be regarded as the foci of a conic. The position of the lines BO, CO will determine when the conic is an ellipse, hyperbola, or parabola.

5824 & 5903. (By Professor Crofton, F.R.S.)—A convex figure whose perimeter is s and area Δ, is thrown at random on a fixed straight band of breadth a traced on a floor; show that the mean value of the area covered by it is $\dfrac{\pi a \Delta}{\pi a + s}$ or $\dfrac{\pi a r^2}{a + 2r}$, when the figure becomes a circle (omitting the cases when the figure does not meet the band).

I. *Solution by* Professor Seitz, M.A.



Let ABCDE be the convex figure, and QRTS the fixed band. Draw PP′ perpendicular to QR, and from B and D, the extreme vertices, draw BF and DG perpendicular to PP′; then GF is the projection of the figure on PP′.

Let $HK = a$, $HF = x_1$, $KF = x_2$, $MN = y_1$, $M'N' = y_2$, $GF = z$, $\angle BIP = \theta$, area ABCNM $= u_1$, and area EDN′M′ $= u_2$.

Then we have $du_1 = y_1\,dx_1$, $du_2 = y_2\,dx_2$, and for the required mean value

$$M = \int_0^{\pi}\left\{\int_{-a}^{z}\Delta\,dx_1 - \int_0^{z} u_1\,dx_1 - \int_0^{z} u_2\,dx_2\right\}d\theta \div \int_0^{\pi}\int_{-a}^{z} d\theta\,dx_1.$$

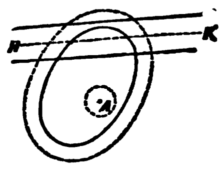
But $$\int_{-a}^{s} \Delta\, dx_1 = (s+a)\,\Delta,$$

and $$\int_0^s u_1\,dx_1 + \int_0^s u_2\,dx_2 = s\Delta - \int_0^s x_1y_1\,dx_1 + \int_0^s x_2y_2\,dx_2 = s\Delta;$$

therefore $\mathrm{M} = \int_0^\pi a\Delta\,d\theta + \int_0^\pi (s+a)\,d\theta$. By the solution of Quest. 5582 (*Reprint*, Vol. XXX., pp. 50, 51), we have $\int_0^\pi s\,d\theta = s$; $\therefore\ \mathrm{M} = \dfrac{\pi a\Delta}{\pi a+s}$.

For the circle, we have $s = 2\pi r$, $\Delta = \pi r^2$, and $\mathrm{M} = \dfrac{\pi a r^2}{a+2r}$.

II. *Solution by the* PROPOSER.



Let M (B) be the mean value of the portion of the band covered by the figure; the chance that a point, taken at random on the figure, shall fall on the band is $p = \dfrac{\mathrm{M\,(B)}}{\Delta}$; which is the same as if the figure were fixed, and the band thrown at random upon it. Now fix the random point at A: then the favourable cases are when HK, the bisector of the band, meets a circle, centre A, radius $\frac{1}{2}a$; and the whole number of cases are when HK meets a curve *parallel to* the given contour S, and at a distance $\frac{1}{2}a$ from it. As the perimeter of this curve is $\mathrm{S} + 2\pi(\frac{1}{2}a)$, we have $p = \dfrac{2\pi(\frac{1}{2}a)}{\mathrm{S}+2\pi\,.\,\frac{1}{2}a}$; being independent of the position of A; therefore $\mathrm{M\,(B)} = \dfrac{\pi a\Delta}{\pi a + \mathrm{S}}$.

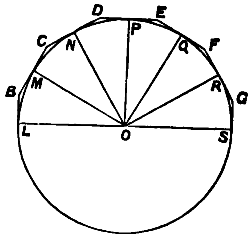
4031. (By Professor SYLVESTER, F.R.S.)—Define *a tree* in its most general form regarded as a complex of relations between any given number of points or ideas.

Solution by the PROPOSER.

(1) A ramification is a system of relations between points such that every point is in relation with some one or more other points, but so that no system of successively related points forms a circuit returning into itself.

(2) A tree is an irreducible ramification, that is, one which is not an aggregate of two or more ramifications.

Such is the *negative definition* of a tree, but we have also a positive *intuition* of a tree, or, say, of arborescence, and from the combination of the positive conception with the negative condition is hatched an illimitable series of hidden consequences constituting the science of the subject. Such is, I believe, the general story of the birth and growth of all pure science; so it is that the *egg is laid.*



5357. (By Professor SYLVESTER, F.R.S.) In the annexed figure, O is the centre of the circle; LB, BC, CD, DE... are any number of tangents to the circle at L, M, N, P, ...; BC is bisected at M, CD at N, DE at P, ..., so that 2LB = BC = CD ... = 2GS. The lines touching the circle, and the radii drawn to the points of contact, are now supposed to be jointed rods moveable freely round each other at either extremity, and it is easily seen that the fan-like figure thus formed becomes a complete linkage, one (that is to say) capable of changing its form in only a single definite series of ways.

1. Prove that, when the fan is opened or closed, the alternate angles LOM, NOP, QOR, ..., always remain equal to each other, as also the alternate angles MON, POQ, ROS, ..., so that the fan may be used to divide any angle into a given number of parts,—as, for example, in the figure above, when the fan is open or closed, the angle LON will always remain one-third of the angle LOS.

2. Imagine planes to become rigidly attached to BC, CD, DE, ..., and prove that, when OL is fixed, the trajectories of points taken in them are unicursal curves of the respective orders.

Solution by W. J. CURRAN SHARP, M.A.

Any figure CMON is such that CMO = CNO in every position; therefore DNO = OMB; and OD bisects NOP, OC bisects MON, and OB bisects LOM; and BOM = DON = FOQ; therefore LOM = NOP = &c., and MON = POQ = &c., and therefore LON = $\frac{1}{n}$ LOS.

This construction derives a trisecting machine from the trisection of a right angle, and in the same way a machine for dividing an angle into n parts may be derived from the division of the right angle into n parts.

6198. (By Dr. HOPKINSON, F.R.S.)—At the Paris Exhibition a freezing machine (Giffard's) was exhibited, consisting of two cylinders, the pistons of which work on to two cranks on the same shaft, driven by an external source of power, and of a large air reservoir which is always maintained at the temperature of the external air. In the first cylinder air is compressed till its pressure is the same as in the reservoir, when valves open and the air passes, as the stroke is completed, into the reservoir. The second and smaller cylinder acts as an engine receiving compressed air from the reservoir for such a portion of the stroke that, being expanded for the remainder of the stroke, it is discharged at atmospheric pressure, but at a lower temperature. If V_1 and V_2 be the volumes of the cylinders, and Π the atmospheric pressure, and if the compression and expansion be supposed adiabatic, prove that the work done during each stroke in the first and second cylinders is

$$\Pi V_1 \frac{\gamma}{\gamma-1} \cdot \frac{V_1-V_2}{V_2}, \quad \Pi \frac{\gamma}{\gamma-1} (V_1-V_2).$$

Solution by J. J. WALKER, M.A.

Let V'_1, V'_2 be the volumes of the parts of the two cylinders occupied by air after compression has ceased and before expansion has commenced, respectively, the pressure in each case being P, as in the reservoir. It is supposed that the mass of air forced into the reservoir from the larger, is equal to that admitted into the smaller cylinder; so that (1) $PV'_2 = \Pi V_1$. The efficient work (W_2) done in V_2 is $W'_2 + W''_2 - W'''_2$; W'_2 being the whole work done by the air admitted before expansion begins: viz.,

$$W'_2 = PV'_2 = \Pi V_1;$$

W''_2 the whole work done during expansion of this air: viz.,

$$W''_2 = \int_{V'_2}^{V_2} p\,dV = c\int_{V'_2}^{V_2} V^{-\gamma}\,dV = \frac{c}{\gamma-1}\left(V^{1-\gamma+1} - V_2^{-\gamma+1}\right)$$
$$= \Pi(\gamma-1)^{-1}(V_1 - V_2),$$

since $c = PV_2'^{\gamma} = \Pi V_2^{\gamma}$ and $PV'_2 = \Pi V_1$; lastly, W'''_2 being the work spent in overcoming the resistance of atmospheric pressure: viz. $W'''_2 = \Pi V_2$. Hence $W_2 = \gamma(\gamma-1)^{-1}\,\Pi(V_1 - V_2)$.

The work (W_1) done by engine-power in the larger cylinder V_1 may be deduced from W_2 by observing that the limiting pressures and mass of gas are the same in both cases; or its value may be obtained independently thus: $W_1 = W''_1 + W'_1 - W'''_1$, W''_1 being the whole work done in compressing the air: viz.,

$$W''_1 = \int_{V'_1}^{V_1} p\,dV = c'\int_{V'_1}^{V_1} V^{-\gamma}dV = (\gamma-1)^{-1}(PV'_1 - \Pi V_1),$$

since $c' = PV_1'^{\gamma} = \Pi V_1^{\gamma}$; W'_1 being the whole work done in forcing the compressed air into the reservoir: viz., $W'_1 = PV'_1$; lastly, W'''_1 being the part of the whole work $W''_1 + W'_1$ done by atmospheric pressure: viz., $W'''_1 = \Pi V_1$. Now, $PV_1'^{\gamma} = \Pi V_1^{\gamma}$ and $PV_2'^{\gamma} = \Pi V_2^{\gamma}$; whence $V'_1 = V_1 V'_2 \div V_2$ and $PV'_1 = V_1 PV'_2 \div V_2 = \Pi V_1^2 \div V_2$, by (1). Thus

$$W_1 = \gamma(\gamma-1)^{-1}(PV'_1 - \Pi V_1) = \gamma(\gamma-1)^{-1}\,\Pi V_1(V_1 - V_2) \div V_2.$$

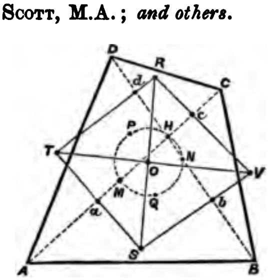
6297. (By C. TAYLOR, M.A.)—The tangent at P to a rectangular hyperbola meets a chord OQ at right angles in R; show that the diameters bisecting the chords PO and PQ bisect the angles between the diameters to P and R.

Solution by Prof. GENESE, M.A.; H. T. GERRANS, B.A.; *and others.*

Let M be the middle point of PO. Then CP, CM are conjugate in direction to RP, MP respectively. Thus angle PCM = angle RPM. But a circle on PO as diameter would pass through R. Therefore MP = MR, and angle MPR = angle MRP. Thus angle PCM = angle MRP, and MPCR lie on a circle. But MP = MR. Therefore MC bisects angle PCR. Again, $RP^2 = RO \cdot RQ$. Therefore QPO is a right angle. The diameters bisecting PO and PQ must therefore be at right angles.

6282. (By Prof. GENESE, M.A.)—ABCD is a quadrilateral, M, N are the middle points of the diagonals AHC, BHD; and the circles HAB, HCD meet in P, and the circles HBC, HDA in Q; prove that the points M, N, H, P, Q lie on one circle.

Solution by F. D. THOMSON, M.A.; Prof. SCOTT, M.A.; *and others.*



Let R, S, T, V be the centres of the circles round HCD, HAB, HAD, HBC: then RTSV is a parallelogram whose sides bisect HA, HB, HC, HD at right angles in the points a, b, c, d.

Let O be the intersection of RS, TV. Then $ac = \frac{1}{2}$ AC = MC, therefore aM = cH; but Oc = Oa, therefore OH = OM. Similarly OH = ON; therefore O is the centre of the circle round HMN. But, since HP is the common chord of two circles, centres R and S, HP is bisected at right angles by RS, therefore OH = OP. Similarly OH = OQ, therefore O is the centre of a circle through HNQMP.

6318. (By J. VENN, M.A.)—A, B, C are set to sort a heap of books; A taking the English political books, and the bound foreign novels; B the bound political works and the English novels, provided they are not political; and C the bound English works, and the unbound political novels. What works will be claimed by both A and B? Will any be claimed by all three?

I. *Solution by* C. J. MONRO, M.A.; R. R. GREY, M.A.; *and others.*

In an obvious notation, say that A, B, C, take

$$ep + be'n, \quad bp + enp', \quad be + b'np.$$

Here ep and bp have bep in common, leaving $b'ep$ (which has nothing in common with enp') and $be'p$ (which has only $be'np$ in common with $be'n$). So A's and B's lots have in common only $bep + be'np$, or bound political books which are either English or novels, including bound *Coningsbys*. It is now plain that C's lot has only bep, bound English political books, in common with A's and B's.

II. *Solution by* H. McCOLL, B.A.

Take one book from the heap, and speaking of this book let α, β, γ respectively denote the statements, A *claims it*, B *claims it*, C *claims it*; and let e, p, b, n respectively denote the statements, It is *English*, It is *political*, It is *bound*, It is a *novel*.

Thus our data are $\alpha = ep + be'n$, $\beta = bp + enp'$, $\gamma = be + b'pn$; and the inference is evident that

$$\alpha\beta = ebp + bpe'n = bp\,(e + e'n) = bp\,(e + n), \quad \alpha\beta\gamma = bep + bepn = bep.$$

That is to say, we infer that both A and B claim every *bound political* work, provided it be *English* or a *novel* (or both); and that A, B, C all three claim the *bound English political* works. The symbol $=$ is here used throughout, for all the implications are true *conversely*. The last equation, for example, $\alpha\beta\gamma = bep$, is equivalent to the double implication

$$(\alpha\beta\gamma : bep)\,(bep : \alpha\beta\gamma).$$

III. *Solution by the* PROPOSER.

Let a, b, c stand respectively for the assignments to the three persons; x, y, z, w for the books which are English, political, bound, and novels (*foreign* being equivalent to *not*-English). Then, in BOOLE's and my notation, the data stand thus:—

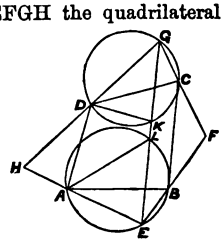
$$a = xy + z\bar{x}w, \quad b = yz + \bar{x}yw, \quad c = xz + \bar{y}zw.$$

Now, community of class membership is determined by (logical) multiplication. Therefore the assignment common to A and B, namely ab, is $xyz + \bar{x}yzw$; and that to A, B, and C, namely abc, is xyz; since $x\bar{x} = 0$, $y\bar{y} = 0$, &c.; which may be read off in the same words as Mr. McCOLL's answer.

6336. (By Rev. H. G. DAY, M.A.)—Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral.

Solution by Dr. LIEBER; J. O'REGAN; H. MURPHY; *and others.*

Let A, B, C, D be the given points, and EFGH the quadrilateral required, the form of which is given. Since we know CD and $\angle$ CGD, the circle round $\triangle$CDG is also known, which may cut EG in K; $\angle$ CDK being $= \angle$ CGK, and the latter being known, the point K is determined. Besides the circle round $\triangle$ AEB is known which cuts EG in L; and $\angle$LAB $= \angle$LEB; consequently L is fixed and the line KL found, which cuts the circles in E and G.



6296. (By E. B. ELLIOTT, M.A.)—A circular city has n gates, n straight streets along radii to these gates, and n circular streets concentric with its wall cutting these at right angles. A person starts from the

centre along any one of the streets and turns whenever possible, but never inwards towards the centre. Show (1) that the whole number of possible routes by which he may thus leave the city is $n2^n$; and that the chance that the gate by which he leaves the city is the one in the direction towards which he originally started is, (2), if n is odd, 2^{1-n}; or (3), if n is even, $2^{1-n}+\dfrac{n!}{2^n(\frac{1}{2}n!)^2}$.

Solution by W. A. WHITWORTH, M.A.; J. HAMMOND, M.A.; *and others.*

1. He may start in n directions, and after that there are n occasions on which he has choice of two routes; therefore the total number of ways is $n2^n$.

2. If n be odd, the only means by which he can leave off in the direction in which he began, is by turning always to the right or always to the left; that is, there are (for each direction in which he may start) two ways; and therefore the chance is 2^{1-n}.

3. If n be even, the same end may be gained, not only as in the last case, but also by turning $\frac{1}{2}n$ times to the right and $\frac{1}{2}n$ times to the left; that is, there are $2+\dfrac{n!}{(\frac{1}{2}n!)^2}$ ways; and therefore the chance is that given in the Question.

5948. (By A. MARTIN.)—A person throws 10 dice, and those that fall "ace up" are taken away. He throws the remainder, and those that fall "ace up" are taken away, and so on, till every die has been turned "ace up." Required the probability that this will be effected in 4 throws.

Solution by W. J. C. SHARP, M.A.; Prof. MATZ, M.A.; *and others.*

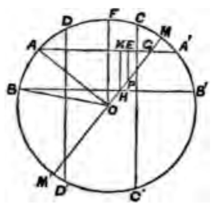
The chance is the same as that of throwing an ace with each die at least once in four throws. Now the chance with each die is

$$\tfrac{1}{6}+\tfrac{5}{36}+\tfrac{25}{216}+\tfrac{125}{1296}=1-(\tfrac{5}{6})^4;$$

hence the chance with all the dice $=[1-(\frac{5}{6})^4]^{10}=$ the chance required, or, more generally, turning ace up with every one of m dice in n throws $=[1-(\frac{5}{6})^n]^m$.

5833. (By COLONEL CLARKE, C.B., F.R.S.)—Two pairs of parallel lines cross a circle so as to form a rectangle within it; prove that (1) the chance that this rectangle contains a given point at the distance δ from the centre of the circle is $\frac{1}{4}(1-\delta^2)^2$; and (2) the chance that it contains a portion of a given diameter of the circle is $\frac{1}{4}(1+\log_e 4)$.

Solution by Prof. Seitz; Prof. Casey; *and others.*



1. Let AA′, BB′ and CC′, DD′ be the two pairs of parallel lines, P the given point, and O the centre of the circle. Draw OF and PE parallel to CC′. Let $OA = 1$, $OP = \delta$, $\angle POF = \theta$, $\angle AOF = \phi$, $\angle BOF = \psi$, $\phi_1 = \sin^{-1}(\delta \sin\theta)$, and $\phi_2 = \cos^{-1}(\delta\cos\theta)$. The number of ways the two pairs of parallel lines can cross the circle so as to form a rectangle within it, AA′ being farther from the centre than BB′, is

$$2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_\phi^{\pi-\phi} AA'^2\, d\theta \sin\phi\, d\phi \sin\psi\, d\psi = 16\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \sin^3\phi\cos\phi\, d\theta\, d\phi = 2\pi.$$

The number of ways the rectangle can be formed, so as to contain the given point, is

$$\int_0^{\frac{1}{2}\pi}\left\{\int_{\phi_1}^{\phi_2}\int_{\phi_2}^{\pi-\phi} 2AE\,.\,A'E \sin\phi\, d\phi \sin\psi\, d\psi + \int_{\pi-\phi_2}^{\pi-\phi_1}\int_{\pi-\phi}^{\phi_2} 2AE\,.\,A'E\sin\phi\, d\phi\sin\psi\, d\psi\right\} d\theta$$

$$= 2\int_0^{\frac{1}{2}\pi}\left\{\int_{\phi_1}^{\phi_2}(\sin^2\phi - \delta^2\sin^2\theta)(\cos\phi + \delta\cos\theta)\sin\phi\, d\phi - \int_{\pi-\phi_2}^{\pi-\phi_1}(\sin^2\phi - \delta^2\sin^2\theta)(\cos\phi+\delta\cos\theta)\sin\phi\, d\phi\right\} d\theta$$

$$= \int_0^{\frac{1}{2}\pi}(1-\delta^2)^2\, d\theta = \tfrac{1}{2}\pi(1-\delta^2)^2.$$

Hence the chance that the rectangle contains the given point is

$$\tfrac{1}{2}\pi(1-\delta^2)^2 \div 2\pi = \tfrac{1}{4}(1-\delta^2)^2.$$

2. Let MM′ be the given diameter intersecting AA′, BB′ in G, H. Draw HK parallel to CC′. Let $\angle MOF = \theta$, $\angle AOF = \phi$, $\angle BOF = \psi$, and $\psi_1 = \cos^{-1}(\sin\phi\cot\theta)$. The number of ways the rectangle can be formed so as to contain a portion of the given diameter is

$$2\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_{\psi_1}^{\pi-\psi_1}(2AK\,.\,A'K + A'K^2)\sin\psi\, d\psi + \int_{\pi-\psi_1}^{\pi-\phi} AA'^2\sin\psi\, d\psi\right\} d\theta\sin\phi\, d\phi$$

$$+ 2\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_\phi^{\pi-\phi}(2AG\,.\,A'K - KG^2)\, d\theta\sin\phi\, d\phi\sin\psi\, d\psi$$

$$= \tfrac{8}{3}\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta(3\cos\phi + \sin\phi\cot\theta)\sin^3\phi\, d\phi + \int_\theta^{\frac{1}{2}\pi}(3\sin^2\phi + 3\sin\phi\cos\phi\tan\theta - 2\cos^2\phi\tan^2\theta)\sin\phi\cos\phi\, d\phi\right\} d\theta$$

$$= \int_0^{\frac{1}{2}\pi}\left\{1 + (\tfrac{1}{2}\pi - \theta)\tan\theta + \theta\cot\theta\right\} d\theta = \int_0^{\frac{1}{2}\pi}(1 + 2\theta\cot\theta)\, d\theta$$

$$= \tfrac{1}{2}\pi - 2\int_0^{\frac{1}{2}\pi}\log\sin\theta\, d\theta = \tfrac{1}{2}\pi(1 + \log_e 4).$$

Hence the chance that the rectangle contains a portion of the given diameter is $\frac{1}{2}\pi(1 + \log_e 4) \div 2\pi = \frac{1}{4}(1 + \log_e 4)$.

6291. (By R. TUCKER, M.A.)—A certain hyperbola is the locus of the mid-points of chords of an ellipse, the perpendicular bisectors of which pass through a fixed point; find the envelop of this hyperbola when the fixed point lies on the auxiliary circle of the ellipse.

Solution by R. KNOWLES, B.A., L.C.P.; J. E. STEGGALL, B.A.; *and others.*

If the equation of the ellipse be $b^2x^2+a^2y^2=a^2b^2$, and (X, Y) the given point, the equation of the hyperbola is obviously $\frac{a^2(X-x)}{x}=\frac{b^2(Y-y)}{y}$; and if $X^2+Y^2=a^2$, we may put $X=a\cos\theta$, $Y=a\sin\theta$, and the equation is $\frac{a^2\cos\theta}{x}-\frac{b^2\sin\theta}{y}=\frac{a^2-b^2}{a}$, and the envelop is

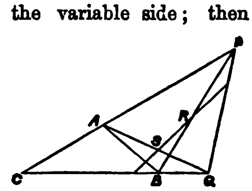
$$\frac{a^4}{x^2}+\frac{b^4}{y^2}=\left(\frac{a^2-b^2}{a}\right)^2,$$

the reciprocal polar of the evolute of the ellipse with respect to the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=a$.

6303. (By D. EDWARDES.)—If a quadrilateral, three of whose sides are fixed, be inscribable in a circle, prove that the line bisecting its diagonals touches a fixed parabola, with respect to which the fixed triangle is self-conjugate, and whose directrix passes through one of the vertices of the triangle.

Solution by Rev. T. W. OPENSHAW, M.A.; Prof. SCOTT, M.A.; *and others.*

Let ABQP be the quadrilateral and PQ the variable side; then $\angle$ BQP $=\pi-$BAP $=$ CAB $=$ constant; so also APQ is constant; therefore PQ moves parallel to itself, and ABC is a fixed triangle.



Let CQ $=q$, CP $=p$;

mid-points of PB, $\frac{1}{2}a$, $\frac{1}{2}p$; of AQ, $\frac{1}{2}q$, $\frac{1}{2}b$;

then the equation to RS is

$$x(p-b)+y(q-a)+\tfrac{1}{2}(ab-pq)=0;$$

but $p=mq$, therefore $\frac{1}{2}mq^2-(mx+y)q+bx+ay-\frac{1}{2}ab=0$;

therefore the envelope is $(mx+y)^2=m(2bx+2ay-ab)$;

the polar of (0, 0) is $bx+ay-ab=0$, *i.e.* AB;

the polar of $(a, 0)$ is $2mxa+may=m\{b(x+a)+ay-ab\}$, or $x=0$, *i.e.* CA.

5909. (By the EDITOR.)—A rod, capable of turning on a given point in its length, is placed at a fixed distance from a centre of force that varies inversely as the square of the distance; find its oscillations.

I. *Solution by* J. J. WALKER, M.A.

Let A be the centre of force, μ its intensity on unit mass at unit distance; C the lower end of the rod BC, oscillating, about the point O, in a given plane through OA. Let α be the cross section, ρ the density, and a, a' the segments OC, OB of the rod; θ the angle it makes with OA at the time t; AD, AE the perpendicular on BC and the line bisecting the angle BAC respectively; b the length of AO. Then the small angles OAC, OAB are equal to $a\theta \div (b-a)$, $a'\theta \div (b+a')$ respectively; whence

$$\angle \text{BAC} = (a+a')\, b\theta \div (b-a)\,(b+a'),$$

and $$\angle \text{OAE} = \left\{(a-a')\,b + 2aa'\right\}\theta \div 2\,(b-a)\,(b+a').$$

Now the resultant attraction on the mass of the rod is along AE, and is equal to $\mu\rho\alpha$ BAC $\div$ AD, and its moment about O is $\mu\rho\alpha$ BAC . OAE $\div \theta$. Hence the angular acceleration is

$$\ddot{\theta} = 3\mu(a+a')\,b\left\{(a-a')\,b+2aa'\right\}\theta \div 2(a^3+a'^3)(b-a)^2(b+a')^2, \text{ say, } \mu k\theta \div k';$$

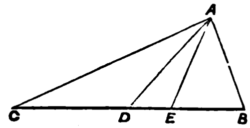
and the length of the equivalent simple pendulum is $l = gk' \div \mu k$. In order that the movement should be stable, $a : a' >$ AC : AB.

In the case of O being the centre, or a half the length, of BC, $k' : k = (b^2-a^2)^2 : 3b$.

In the case of the point O coinciding with the end B, or a being the whole length of BC, $k' : k = 2a\,(b-a)^2 : 3$; which, when A is very distant, and $\mu \div b^2 = g$, gives ($a \div b$ vanishing) $2a \div 3$ as the length of the simple pendulum equivalent to a rod of length a oscillating about one end—as it should.

II. *Solution by* G. S. CARR, B.A.

Let A be the centre of force, BC $= 2b$ the rod, D its centre, AE the bisector of $\angle$BAC, AD $= a$, $\angle$BDA $= \theta$.



The resultant attraction of A upon BC acts along EA, and is $= 2\mu \sin \text{BAD} \div a \sin\theta$ (TODHUNTER'S *Statics*, Art. 204). The moment of this force about D is

$$2\mu \sin \text{BAD} \sin \text{DAE} \div \sin\theta$$
$$= \mu\,(\cos \text{DAC} - \cos \text{DAB}) \div \sin\theta.$$

Let $\quad x = \tan \text{DAB} = \dfrac{b\sin\theta}{a-b\cos\theta},\quad y = \tan \text{DAC} = \dfrac{b\sin\theta}{a+b\cos\theta},$

then the above becomes $\mu\left\{(1+y^2)^{-\frac12} - (1+x^2)^{-\frac12}\right\} \div \sin\theta$.

Expand this, then we may write the equation of motion as follows:—

$$k^2\frac{d^2\theta}{dt^2} + \frac{\mu}{\sin\theta}\left\{\tfrac12\,(x^2-y^2) - \tfrac38\,(x^4-y^4) + \&\text{c.}\right\} = 0.$$

For small oscillations about the position of equilibrium, for which $\theta = 0$, we may neglect the terms after (x^2-y^2); hence, by putting $k^2 = \frac13 b^2$,

$$x = \frac{b\theta}{a-b},\quad y = \frac{b\theta}{a+b},\quad \text{we obtain } \frac{d^2\theta}{dt^2} + \frac{6\mu ab}{(a^2-b^2)^2}\,\theta = 0;$$

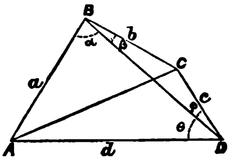
and the time of oscillation $= 2\pi\,(a^2-b^2) \div (6\mu ab)^{\frac12}$.

If a be very great, $\quad t = 2\pi\,(a^3-2ab^2)^{\frac12} \div (6\mu b)^{\frac12}$,
increasing with the distance.

6302. (By J. YOUNG, B.A.)—If AB, AD be the greater sides of a quadrilateral ABCD inscribed in a circle, prove that

$$\frac{\sin(A+B)}{\sin(A+D)} = \frac{AB^2-CD^2}{AD^2-BC^2}.$$

Solution by G. EASTWOOD, M.A.; E. B. ELLIOTT, M.A.; *and others.*



In the annexed diagram, we have

$\sin(A+B) = \sin(\pi-\alpha-\theta+\alpha+\phi) = \sin(\theta-\phi)$;

similarly $\sin(A+D) = \sin(\alpha-\beta)$;

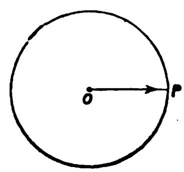
also $\sin(\theta+\phi) = \sin(\alpha+\beta)$;

$$\therefore\ \frac{\sin(A+B)}{\sin(A+D)} = \frac{\sin^2\theta-\sin^2\phi}{\sin^2\alpha-\sin^2\beta} = \frac{AB^2-CD^2}{AD^2-BC^2}.$$

6299. (By C. LEUDESDORF, M.A.)—A circular hoop of radius a, lying on a smooth horizontal plane, is rotating with angular velocity ω about an axis through its centre perpendicular to its plane, and a particle is projected from the centre of the hoop with velocity v against its inner surface, supposed perfectly rough; show that, if e be the coefficient of elasticity, and n the ratio of the mass of the particle to that of the hoop, the particle will strike the hoop again after a time

$$\frac{2(2n+1)^2 aev}{(2n+1)^2e^2v^2+(n+1)^2a^2\omega^2}.$$

Solution by J. E. A. STEGGALL, B.A.; D. EDWARDES; *and others.*



Let P be the particle at the instant of striking. To find the velocity along OP after the blow, we have,

if bodies were inelastic, $nv=(n+1)V$, $nV = \dfrac{nv}{n+1}$;

but bodies are elastic; and therefore, after the blow, the velocities of hoop and particle are

$$\frac{nv}{n+1}(1+e),\ \frac{nv}{n+1}-\frac{ev}{n+1}\ \ldots\ldots\ldots\ldots(1).$$

Also, Ω being the angular rotation of the hoop after the blow, U the velocity of the centre perpendicular to OP, and W the velocity of the particle, we have $W = U + a\Omega$ (through perfect roughness),

$a\Omega + nW = a\omega$ (by moments), and $U + nW = 0$ (no external forces);

therefore $$W = \frac{a\omega}{2n+1},\quad U = -\frac{na\omega}{2n+1}\ \ldots\ldots\ldots\ldots\ldots\ldots(2);$$

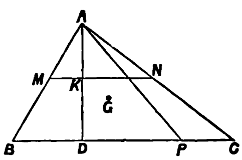
therefore relative velocity of particle in this direction $= \dfrac{a\omega\,(n+1)}{2n+1}$; and by (1), in direction OP, $= -ev$.

$$\text{Now} \qquad \text{time} = \frac{\text{chord}}{\text{resultant vel.}} = \frac{2a\cos\theta}{\text{result. vel.}}$$

$$= \frac{2a\,(\text{vel. along PO})}{(\text{result. vel.})^2} = \frac{2\,aev\,(2n+1)^2}{e^2v^2\,(2n+1)^2 + (n+1)^2a^2\,\omega^2}.$$

6275. (By Prof. CROFTON, F.R.S.)—Show that the mean distance of the vertex of a triangle from all points in the area is equal to its distance from the centre of gravity, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centre of gravity in a direction parallel to the other,—the axis of the parabola being parallel to the base.

I. *Solution by* G. HEPPEL, M.A.; Prof. MATZ, M.A.; *and others.*



Take A as origin, AD as positive axis of y. Let AD $= h$, DC $= l$, DB $= m$, DP $= z$, and let G be the centre of gravity. Let BC $= l+m = a$, then the mean distance of A from BC $= \dfrac{1}{a}\displaystyle\int_{-m}^{l} (h^2+z^2)^{\frac{1}{2}}\,dz = p$. And, if AK $= y$, mean distance from MN $= \dfrac{yp}{h}$; therefore, if P = mean distance from all points in the triangle,

$$\text{P} = \int_0^h \frac{yp}{h}\cdot\frac{ya}{h}\cdot\frac{2}{ah}\,dy = \tfrac{2}{3}\,p = \frac{2}{3a}\int_{-m}^{l}(h^2+z^2)^{\frac{1}{2}}\,dz, \text{ or, putting } z^2 = 3ax,$$

we have
$$\text{P} = \int_{\frac{m^2}{3a}}^{\frac{l^2}{3a}} \left(\frac{h^2}{3ax}+1\right)^{\frac{1}{2}} dx.$$

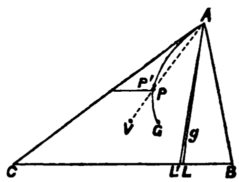
But the length of an arc of any parabola $(y-b)^2 = 4d\,(x-c)$, taken between the limits $x=e$ and $x=f$, is $\displaystyle\int_f^e \left(\frac{d+x-c}{x-c}\right)^{\frac{1}{2}} dx$; hence P = length of such an arc, the latus rectum being $\dfrac{4h^2}{3a}$, and the limits of integration $e = \dfrac{l^2}{3a}+c$ and $f = \dfrac{m^2}{3a}+c$. Let $f=0$, then $c = -\dfrac{m^2}{3a}$ and $e = \frac{1}{3}\,(l-m)$, and the parabola is
$$(y-b)^2 = \frac{4h^2}{3a}\cdot\left(x+\frac{m^2}{3a}\right).$$

Take $b = \dfrac{2hm}{3a}$, and the equation becomes $y^2 - \dfrac{4hmy}{3a} = \dfrac{4h^2x}{3a}$, which passes through the vertex where $x = f = 0$. If $x = e = \frac{1}{3}\,(l-m)$, we get $y = \frac{2}{3}h$; therefore the parabola passes through the centre of gravity.

For intersection with AB, put $x = -mh^{-1}y$, and we get $y^2 = 0$; therefore AB is a tangent.

The equation to a parallel to AC through G is $y - \frac{2}{3}h = \frac{h}{l}\{x - \frac{1}{3}(l-m)\}$; and, for the intersection of this with the parabola, we get $(y - \frac{2}{3}h)^2 = 0$; therefore this parallel is also a tangent.

II. *Solution by the* Proposer.



Conceive a line AL to turn continuously round A, from the position AB to AC; and, as it revolves, imagine that the whole mass of the triangle *to the right* of AL is collected into the vertex A. The centre of gravity will thus move from G to A, along a certain curve. As the line moves from the position AL to the consecutive one AL′, the centre moves from the position P to P′ on its locus; PP′ being parallel to AL; also $PP' = \frac{ALL'}{ABC} \cdot \frac{2}{3}AL$, as the mass ALL′ has been transferred to A, its centre of gravity g having moved over gA.

Hence, if $ABC = \Delta$, we have arc $AG = \frac{1}{\Delta}\Sigma\,(ALL' \times \frac{2}{3}AL)$ for all positions of AL. Now, to find the mean distance of A from all points in the triangle, we have to multiply every element of the area by its distance from A, and divide by Δ; but the portion of this sum due to elements on the triangle ALL′ is $ALL' \times \frac{2}{3}AL$, therefore mean distance = arc AG.

It is easy to see that the curve AG is a parabola, as when the centre of the whole mass is at P, it must be in a line with A and the centre V of the triangle ACL; so that, if A be the origin, and the axes be AC and a parallel to CB, $2x : y = CL : b$; also $AP : AV = CL : a$; whence $CL = \frac{3ay}{2b}$; therefore $y^2 = \frac{4b^2}{3a}x$.

The mean distance of a given point from any plane area is equal to the length of the path of the centre of gravity of the mass, supposed continuously transferred to the given point as above.

"At Random" Again. By C. J. Monro, M.A.

The proper meaning of distribution "at random" has been largely discussed in various volumes of the *Reprints*, but perhaps not exhaustively. Questions of science have not, I venture to think, been sufficiently discriminated from questions of the convenient use of language: the difference has not always been observed in practice, and I do not find that at any stage of the controversy it has been laid down in theory. It appears to me

that the phrase covers two quite different things; one a strictly scientific notion, the other a more or less appropriate convention.

It is universally agreed that if a point is taken at random in space it is equally likely to be in any two portions of equal volume. What is the justification of this? Simply the fact that (in Euclidean space of course, and with a limitation to be pointed out below) it is impossible to lay down any other condition which is not either self contradictory, as that different probabilities should belong to equal volumes of different form, or insufficiently general, as that different probabilities should belong to equal volumes at different distances from St. Paul's. This is an absolutely definite mathematical notion, and it is probably what they have in view who say that, things taken at random are taken "according to no law" (*Reprint*, VI., 82, VII., 83, 84): but as I do not pretend to attach a definition to a received term, I shall use another word, *indifference*, and say that a locus is placed or given *indifferently* when it is impossible to indicate consistently where or how it is placed except by reference to things not implied in such determination as it has. Cases of indifference are those of a point in space, in an infinite plane, on a sphere, on a right cylinder, on an infinite straight line, on the circumference of a circle, on a helix. So of an infinite straight line or plane in space, or an infinite straight line in an infinite plane. If a straight line or plane passes through a given point, its disposition is indifferent: given the disposition, the direction or aspect is indifferent. A sphere or circle of given size is indifferent in other respects. A locus of given size or shape is indifferent as to the place of any one of its points and the directions of any two not parallel straight lines in it. But two points cannot be taken indifferently, because in their determination as "two points" is implied a mutual distance; and with reference to this as greater or less, it is possible to indicate consistently how the points are placed; we may suppose them more likely to be nearer or further apart. It is true there is no natural unit of distance in the Euclidean world; but there is a natural distinction between directions of change in distance; increase is absolutely different from decrease. In general, a locus cannot be indifferent in size or shape. A point cannot be taken indifferently within a bounded range, because it may be more likely to be nearer or further from the boundary; and in connexion with certain problems it may be interesting to observe a limitation alluded to above, that loci cannot be taken indifferently in an infinite space which is conceived as other than spherical or circular, because there may be different probabilities in different assignable directions. An abstract quantity cannot be given indifferently, because it, or the real quantities in it, may be more likely to be nearer or further from naught: but time may, because we know of no natural zero-point of time.

Without suggesting that the term "at random" should be restricted to this meaning, I maintain that when it has not this meaning it has no scientific meaning; and, in any particular case in which this meaning is inadmissible, you cannot say that such and such a meaning is right or wrong, but only that it is more or less convenient. There are indeed simple and general cases in which the convenience is great, and the convention made is easy and obvious. Thus, although two points cannot be taken indifferently, we may agree that a second shall be considered as taken independently of the first. So when a point is said to be taken at random within a given range, it is of course easily understood that the taking differs from that made in unbounded space (as an infinite plane or the whole surface of a sphere) only by the exclusion of exterior points. And the case of abstract magnitude may be likened to that of concrete measurement from merely

arbitrary zeros. But more convention is required if we are to speak of taking at random the size or shape of a locus, and perhaps the intention ought to be specified in each case. It is unreasonable to select some conspicuous constant or constants, as the radius of a circle or the axes of an ellipse, and suppose uniform variation of the same. Why not the square of a selected constant (which indeed one can find a reason for, in the case of a circle in a plane, but only such a reason as would require the cube of the radius of a sphere), and why not any selected function of it? If any general convention is to be made, there is one which seems less arbitrary than any other, namely, to suppose such variations equally probable as affect uniformly the probability that a point taken indifferently will be enclosed by the locus in question; that is, to suppose uniform variation of the content, the length, area, or volume.* If, in course of such variation, successive boundaries intersect, the whole of the differential content between them must of course nevertheless be taken with the same sign: it will therefore be seen that this convention would give a condition for variation in mere position, and the same condition as indifferent distribution would give. The same convention may be applied to the distribution of lines or surfaces within ranges of space, as of chords in a circle. If I was obliged to give a meaning to the term "random chord," I should prefer the chord which cut off a uniformly varying area, so that $\pi^{-1}(\phi - \sin\phi)$ would be the probability that it subtended at the centre an angle less than ϕ. But I submit that the term is too arbitrary, and that the discussions about it in the *Reprints* are a good illustration of the misuse of the word *random*. Several meanings have been assigned, different from that just suggested. One, depending on uniform distribution of the extremities, has been recommended or rather laid down in language that would not want emphasis if applied to the doctrine of the conservation of energy, but without any scientific justification that I can find. The reason offered by Mr. Woolhouse (*Reprint*, V, 110, Solution of Question 1894) does not appear to me to meet the only difficulty, namely, why the distribution of the chords is to be governed by the distribution of their extremities. Miss Blackwood's argument (XXIX, 19) that we do in fact define a chord by reference to its extremities, is precisely to the point, and that is much; but I venture to think the unanswerable answer had been given already by Mr. Godfray (VII, 66), that this definition is not the only possible one. The utmost that can be said is that it is the usual one, the most ancient one it may be,† or the most elegant; but to decide on such grounds is not science, but literary criticism.

Closely connected with random distribution is the difficult subject (discussed in XXIII, 68—70; XXXI, 100—103); if a bag contains a given number of balls about which nothing is known except that they are black and white, whether "every possible partition of the number" or "every possible partition of the *things* (treated as 'different')" is *a priori* equally likely. I call it a difficult subject, because any full discussion of it goes to the root of the mathematical theory of probability, and I would ask any person who is quite content with the latter assumption, what he thinks

* This convention would give at once without calculation the answer "an even chance" to the Editor's Question 6016.

† Is it, though? It seems implied by the metaphor contained in the word. But I find neither the word nor the metaphor in Euclid. It may be said however that ἐνταθῆναι, used in Plato's Menon, 86, 87, of inscribing a triangle of given area, expresses the same metaphor.

of the following bit of dialogue:—"You are absolutely ignorant how these things are divided between the two categories; I must not ask you to tell me even approximately the probability that they are divided about equally." "Yes, *certainty*, if there are enough of them." But whatever the difficulties of the subject, I cannot think there is any difficulty in answering the bare question: the answer is, neither the one nor the other; the probability of any specified partition is indeterminate. The case is not one of indifference, as is by hypothesis the question whether the white or the black are the more numerous; therefore the equal probability of the possible partitions is on both interpretations neither axiomatic nor demonstrable. The point is not as new as at least one writer seems to think: BOOLE raised it in the 20th chapter of his *Laws of Thought*. In fact, he connected it with the subject of this paper, but in a way in which, as the reader will easily understand, I do not see how to follow him.

6161. (By Prof. GENESE, M.A.)—If a straight line through the focus of a conic meet the tangents at the ends of the transverse axis in P, Q, prove that the circle whose diameter is PQ will touch the conic.

Solution by W. E. WRIGHT, B.A.; G. TURRIFF, M.A.; *and others.*

Let the two straight lines be $x^2 = a^2$, and the focus $(c, 0)$, and let the straight line make an angle θ with the axis; then the equation of the circle on PQ as diameters will be

$$(x^2-a^2)+[y+(c-a)\tan\theta]\,[y+(a+c)\tan\theta] = 0,$$

or
$$x^2+y^2-a^2+2cy\tan\theta+(c^2-a^2)\tan^2\theta = 0,$$

and the envelop is $c^2y^2+(a^2-c^2)(x^2+y^2-a^2) = 0$, or $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-c^2} = 1$,

a conic to which the straight lines $x^2 = a^2$ are the tangents at the ends of the transverse axis, and the point $(c, 0)$ is a focus. The conic is an ellipse if $a > c$, and an hyperbola if $a < c$. For real contact, the line $cy+(c^2-a^2)\tan\theta = 0$ must meet the conic in real points,

therefore
$$\frac{a^2-c^2}{c^2}\tan^2\theta < 1, \quad \tan^2\theta < \frac{c^2}{a^2-c^2},$$

which is always true in the hyperbola; but in the ellipse the line must lie between the two straight lines joining the focus to the ends of the minor axis.

5982. (By S. TEBAY, B.A.)—Find three numbers such that, if the cube of their sum be increased by the product of every two of them, the three sums shall be cubes.

Solution by the Rev. Dr. Knisely; H. Pollexfen, B.A.; *and others.*

Let ax, ay, az be the numbers, and ax_1, ay_1, az_1 the roots of the respective cubes; then, if we suppose $x+y+z=1$, we shall have

$$a^3+a^2xy = a^3x_1^3,\ a^3+a^2x_1z_1 = a^3y_1^3,\ a^3+a^2y_1z_1 = a^3z_1^3;$$

and, putting $x_1^3-1=p$, $y_1^3-1=q$, $z_1^3-1=r$, and $Q=pq+pr+qr$, we find $a=\frac{xy}{p}=\frac{xz}{q}=\frac{yz}{r}$; $x=\frac{pq}{Q}$, $y=\frac{pr}{Q}$, $z=\frac{qr}{Q}$; therefore $a=\frac{pqr}{Q^2}$; and the three numbers and the roots of the three cubes are, respectively,

$$\frac{p^2q^2r}{Q^3},\ \frac{p^2qr^2}{Q^3},\ \frac{pq^2r^2}{Q^3};\quad \frac{pqr}{Q^2}x_1,\ \frac{pqr}{Q^2}y_1,\ \frac{pqr}{Q^2}z_1.$$

[In *numbers*, one correspondent finds $\frac{63\,.\,234}{323^3}$, $\frac{26\,.\,234}{323^3}$, $\frac{234^2}{323^3}$.]

5914. (By R. Tucker, M.A.)—How must a ball be struck from the edge of a circular billiard table, so that, after n reflexions, it may come back to the point of departure? Discuss the question for other positions of the ball.

Solution by W. J. C. Sharp, M.A.

If α_1, α_2, &c. be the successive angles made by the path with the radii at the points of impact, $\tan\alpha_1 = \epsilon\tan\alpha_2 = \epsilon^2\tan\alpha_3$, &c. ....(1).
Now, in the first case, $\alpha_1+\alpha_2+\ldots+\alpha_n = \frac{1}{2}(n-2)\pi$, therefore $\tan(\Sigma\alpha)=0$ or ∞ according as n is even or odd. And, by substituting their values from (1) for $\tan\alpha_2$, $\tan\alpha_3$, &c., an $(n-1)$-ic equation is obtained to determine α_1. If the point be not on the circumference, but at a distance $a<r$ from the centre, and if

$$\sin\theta = \frac{r}{r-a}\sin\alpha_1,\ \text{and}\ \sin\phi = \frac{r}{r-a}\sin\alpha_n\ \ldots\ldots\ldots\ldots(2),$$

$$\alpha_1+2\alpha_2+2\alpha_3+\ldots 2\alpha_{n-1}+\alpha_n+\theta+\phi = (n-2)\pi,$$

therefore $\tan(\alpha_1+2\alpha_2+2\alpha_3+\ldots+\alpha_n+\theta+\phi)=0,$

from which, by (1) and (2), an equation to determine $\tan\alpha_1$ can be deduced.

6147. (By E. W. Symons, B.A.)—If A, B, C be the angles of a spheric triangle, and x, y, z any real quantities, prove that

$$x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C\ (=\Pi\ \text{say})\ \text{is positive.}$$

Solution by Prof. Wolstenholme; W. Wilkins, B.A.; *and others.*

$$\Pi \equiv (x-y\cos C-z\cos B)^2+\left\{y\sin C-z\operatorname{cosec} C\ (\cos A+\cos B\cos C)\right\}^2$$
$$+z^2\operatorname{cosec}^2 C\left\{\sin^2 B\sin^2 C-(\cos A+\cos B\cos C)\right\}^2;$$

and hence (for real values of x, y, z) Π is always positive, if

$$\sin^2 B \sin^2 C - (\cos A + \cos B \cos C)^2$$

is positive, or if $[\cos A + \cos (B-C)]\,[-\cos A - \cos (B+C)]$ be positive, *i.e.*, if $\cos S + \cos (S-A) \cos (S-B) \cos (S-C)$ be negative, where $2S = A + B + C$, which is always true when A, B, C are angles of a spherical triangle, S being $> \frac{1}{2}\pi$, and $S-A, S-B, S-C < \frac{1}{2}\pi$.

Otherwise: $\Pi \equiv X^2 + \ldots + 2yz \cos (\pi - A) + \ldots \equiv$ square of distance from origin of (x, y, z) referred to oblique axes inclined at angles $\pi - A$, $\pi - B$, $\pi - C$, and is therefore always positive. [The question is an extension of Problem 513, in the 2nd edition of Prof. WOLSTENHOLME's *Problems*.]

6300. (By J. L. McKENZIE, B.A.)—In any triangle that is not equilateral, if Δ be the area, and l, m, n be the lines bisecting the angles and terminated by the opposite sides, prove that

$$lmn > \frac{64\Delta^3}{(b+c)(c+a)(a+b)}.$$

Solution by H. MURPHY; J. O'REGAN; *and others.*

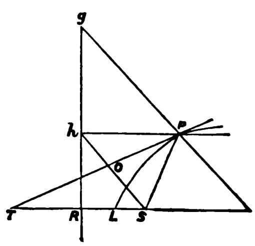
$2\Delta = (b+c)\,l \sin \frac{1}{2}A = (c+a) \sin \frac{1}{2}B = (a+b) \sin \frac{1}{2}C$; therefore

$$lmn = \frac{8\Delta^3}{(b+c)(c+a)(a+b) \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} > \frac{64\Delta^3}{(b+c)(c+a)(a+b)},$$

since (WOLSTENHOLME's *Problems*, Ex. 517) $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C < \frac{1}{8}$.

6304. (By J. HAMMOND, M.A.)—If at any point of a parabola the normal be produced to meet the directrix in g, prove that the radius of curvature at P is double the length of Pg.

Solution by C. BICKERDIKE, L.C.P.; W. E. WRIGHT, B.A.; *and others.*



$Pg : Ph = Ph : Oh\,(SO)$,

therefore $SO \,.\, Pg = Ph^2 = PS^2$.

But $R = \dfrac{2SP^2}{SO} = \dfrac{2SO \,.\, Pg}{SO}$

$= 2Pg$.

6258. (By H. McColl, B.A.)—Assuming as an established law that the event C happens whenever A happens with X, and also whenever B happens without X; that C never happens when B happens with X, nor when A happens without X; that A happens on an average 3 times out of 5, and B once out of 3 times; that A, when it happens, is followed by C once in 10 times (on an average), and B by C 11 times in 20. Show that X will happen on an average between 63 and 83 times in 300.

Solution by the Proposer.

Definitions.—The symbol x_a denotes the chance that the statement x is true *on the assumption that a is true.* The symbol ϵ is an equivalent for the symbol 1, and denotes any statement whose truth is assumed *throughout the whole of an investigation.* Hence, the symbol x simply denotes *the chance that x is true.*

The data of the question are $a_\epsilon = \frac{3}{5}$, $b_\epsilon = \frac{1}{3}$, $c_a = \frac{1}{10}$, $c_b = \frac{11}{20}$, with the two implications $ax + bx' : c$ and $bx + ax' : c'$; and we are required to find x_ϵ, or its limits.

Solving the given implications with respect to x and x', we get

$$bc' + ac : x, \text{ and } ac' + bc : x'.$$

Hence $(bc' + ac)_\epsilon$ is an inferior limit to x_ϵ, and $1-(ac' + bc)_\epsilon$ is a superior limit.

Now, $(bc' + ac)_\epsilon = (bc')_\epsilon + (ac)_\epsilon = b_\epsilon c'_b + a_\epsilon c_a = \frac{3}{20} + \frac{3}{50} = \frac{63}{300}$,

and $1-(ac' + bc)_\epsilon = 1 - a_\epsilon c'_a - b_\epsilon c_b = 1 - \frac{27}{50} - \frac{11}{60} = \frac{83}{300}$.

Thus the required chance is between $\frac{63}{300}$ and $\frac{83}{300}$.

6031. (By the Editor.)—What is the simplest possible six-valued function of six symbols 123456?

Solution by the Rev. T. P. Kirkman, M.A., F.R.S.

The six values of the 6-valued function made with 123456 are $(11 = 1^2)$

$$\begin{aligned} &1^22^234 + 5^21^242 + 6^24^231 + 2^24^215 + 5^22^246 \\ &+ 5^24^261 + 6^22^235 + 2^21^256 + 6^23\,51 + 1^23^264 \\ &+ 3^21^225 + 3^26^224 + 5^23^226 + 2^26^214 + 5^26^221 \\ &+ 6^25^243 + 1^24^253 + 1^26^245 + 5^24^232 + 3^24^212 \\ &+ 2^23^216 + 5^22^213 + 4^21^262 + 3\,5^241 + 5^21^263 \\ &+ 4^26^252 + 4^23^265 + 3^22^254 + 6^21^223 + 4^22^236 = A_1, \end{aligned}$$

which is the simplest possible algebraic function of six letters that has six values. The five values are got by putting 1 for 2, and 2 for 1, 13 for 31, 14 for 41, 15 for 51, 16 for 61; thus,

$$2^21^234 + 5^22^241, \text{\&c.} = A_2,\ 3^22^214 + 5^23^242, \text{\&c.} = A_3,\ 4^22^231 + 5^24^212, \text{\&c.} = A_4,$$
$$5^22^234 + 1^25^242, \text{\&c.} = A_5,\quad 6^22^234 + 5^26^242, \text{\&c.} = A_6;$$

and no substitution—*e.g.*, 216534 or 625341—will give a new value, but merely change the order of terms in some value. Of course the first terms of A_1 and A_2 are the same.

5113. (By the EDITOR.)—124, 154, 156, 163, 234, 132, 473, 456, 467, 367, and 124, 143, 126, 234, 167, 523, 173, 562, 567, 573 are two sets of ten triplets each made with seven elements, so that every duad employed occurs twice in either system. If we change them by any substitution, as 2135746—which means, put 2 for 1, 1 for 2, 3 for 3, 5 for 4, 7 for 5, 4 for 6, and 6 for 7—the first becomes 215, 275, 274, 243, 135, 231, 563, 574, 546, 346, which is similar to the first; but no substitution can make the first into the second, or the second into the first; so that these two are dissimilar sets. Find the entire number of dissimilar sets.

Solution by the Rev. T. P. KIRKMAN, M.A., F.R.S.

The five sets are as follows:—

124, 154, 156, 163, 234, 132, 473, 456, 467, 367;
124, 143, 126, 234, 167, 523, 173, 562, 567, 573;
126, 267, 743, 732, 451, 745, 567, 165, 123, 341;
123, 236, 674, 275, 762, 574, 541, 134, 346, 152;
254, 432, 123, 341, 265, 541, 675, 571, 162, 167;

which are the five Δ-faced 10-edra, of four different symmetries, the figures denoting the summits. My method shows that no more are possible. You may construct any one on any triangle, writing four figures inside it.

6209. (By C. LEUDESDORF, M.A.)—A uniform inextensible string AB of length a is lying in a smooth horizontal plane in the form of the spiral whose intrinsic equation is $\log \frac{s}{c} = \frac{3\phi}{2}$, A being the point where $\phi = 0$. It is acted on by an impulsive tension at A: find the impulsive tension at any point P of the string; and show that, if P begin to move in a direction making an angle θ with the tangent at P to the spiral, then

$$\text{arc AP} = a\left(\frac{\tan\theta + 2}{2 - 4\tan\theta}\right).$$

Solution by Prof. WOLSTENHOLME, M.A.; Prof. GENESE, M.A.; *and others.*

The intrinsic equation of an equiangular spiral being $s = c\epsilon^{\phi \cot \alpha}$, let T, T + δT be the impulsive tensions at P, Q, arc to P = s, arc PQ = δs, u, v the tangential and normal velocities instantaneously communicated to PQ, $\kappa\delta s$ the mass of PQ; then we have

$$\kappa\delta s \,.\, u = (T + \delta T)\cos\delta\phi - T, \text{ or } \kappa u = \frac{dT}{ds},$$

$$\kappa\delta s \,.\, v = (T + \delta T)\sin\delta\phi, \qquad \text{or } \kappa v = T\frac{d\phi}{ds},$$

and, since PQ is unchanged in length,

$$\frac{du}{ds} = v\frac{d\phi}{ds}, \text{ therefore } \frac{d^2T}{ds^2} = T\left(\frac{d\phi}{ds}\right)^2;$$

and, eliminating s by the given equation, $\frac{d^2T}{d\phi^2} - \frac{dT}{d\phi}\cot\alpha = T$; whence $T = A_1\epsilon^{m_1\phi} + A_2\epsilon^{m_2\phi}$, where m_1, m_2 are roots of the equation $z^2 - z\cot\alpha = 1$,

and $$\tan\theta = \frac{v}{u} = \frac{T\,d\phi}{dT} = \frac{A_1\epsilon^{m_1\phi} + A_2\epsilon^{m_2\phi}}{m_1A_1\epsilon^{m_1\phi} + m_2A_2\epsilon^{m_2\phi}},$$

$$A_1\epsilon^{m_1\phi}(m_1\tan\theta - 1) = A_2\epsilon^{m_2\phi}(1 - m_2\tan\theta);$$

therefore $$\frac{A_1}{A_2}\left(\frac{s}{c}\right)^{\frac{m_1-m_2}{\cot\alpha}} = \frac{1-m_2\tan Q}{m_1\tan\theta - 1};$$

and, since the tension at the free end is 0, $\frac{A_1}{A_2}\left(\frac{a}{c}\right)^{(m_1-m_2)\tan\alpha} = -1$,

or $\left(\frac{s}{a}\right)^{(m_1-m_2)\tan\alpha} = \frac{m_2\tan\theta - 1}{m_1\tan\theta - 1}$, *i.e.*, arc AP $= a\left(\frac{m_2\tan\theta - 1}{m_1\tan\theta - 1}\right)^{\frac{\cot\alpha}{m_1-m_2}}$;

and when $\cot\alpha = \frac{3}{2}$, $m_1 = 2$, $m_2 = -\frac{1}{2}$, giving arc AP $= a\left(\frac{\tan\theta + 2}{2 - 4\tan\theta}\right)$.

Thus the general result is $\quad$ arc AP $= a\left(\frac{m + \tan\theta}{m - m^2\tan\theta}\right)^{\frac{m^2-1}{m^2+1}}$,

where $m - m^{-1} = \cot\alpha$; or where the intrinsic equation of the spiral is

$$\log\frac{s}{c} = (m - m^{-1})\cot\phi;$$

and a is the length of the arc measured from the pole (not from the point where $\phi = 0$, but where $\phi = -\infty$). Also, s (or AP) is measured from the pole of the spiral, since, by the given equation, $s = 0$ when $\phi = -\infty$.

1588. (By Professor SYLVESTER, F.R.S.)—Solve the equation

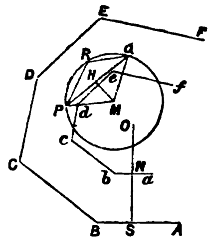
$$ax^5 + 5bx^4 + 10cax^3 + 10cbx^2 + 5c^2ax + c^2b = 0.$$

Solution by the PROPOSER.

Let $x + c^{\frac{1}{2}} = u$, $x - c^{\frac{1}{2}} = v$, $A + B = a$, $(A - B) = bc^{-\frac{1}{2}}$; then $Au^5 + Bv^5 = 0$. Hence $\frac{u}{v}$, and consequently x, becomes known.

6223. (By Professor SEITZ, M.A.)—Within the perimeter ($= s$) of an n-gon circumscriptible about a circle of radius r, three points are taken at random; prove that (1) the chance that the circle through these three points will lie wholly within the polygon is $\frac{2}{3}\pi^2r^2s^{-2}$; and (2) when the polygon becomes a circle, the chance (which is then that of the EDITOR's Question 1843) is $\frac{2}{3}$.

Solution by the PROPOSER.



Let ABCDEF ... be the polygon; PQR the triangle formed by joining the three random points P, Q, R; O the centre of the inscribed circle of the polygon; and M the centre of the circle circumscribing PQR. Draw the polygon *abcdef* ..., making its sides parallel to those of ABCDEF..., and at a distance from them equal to MP; and draw ONS and MH perpendicular to AB and PQ.

Now, while PQ is given in length and direction, and the angle PRQ is given, if MP is less than OS, the area of the polygon *abcdef* ... represents the number of ways the three points can be taken, so that the circle circumscribing the triangle will lie wholly within the given polygon.

Let PQ $= 2x$, OS $= r$, perimeter of ABCDEF ... $= s$, area of segment PRQ $= t$, area of sector PMQ $= v$, area of triangle PMQ $= u$, area of polygon ABCDEF ... $= \Delta$, $\angle$PMP $= \theta$, $\phi = \sin^{-1}(x+r)$, and = the angle which PQ makes with some fixed line.

Then we have PM $= x \operatorname{cosec} \theta$, ON $= r - x \operatorname{cosec} \theta$,

$$\text{area } abcdef = \frac{\Delta}{r^2}(r - x \operatorname{cosec} \theta)^2,\quad v = \theta x^2 \operatorname{cosec}^2 \theta,\quad u = x^2 \cot \theta,$$

$$t = v - u, \text{ and } dt = dv - du = 2x^2 \operatorname{cosec}^2 \theta\,(1 - \theta \cot \theta)\,d\theta.$$

An element of the polygon at Q is $4x\,dx\,d\psi$, or $4r^2 \sin\phi \cos\phi\,d\phi\,d\psi$, and at R it is dt; the limits of x are 0 and r, and of ϕ, 0 and $\frac{1}{2}\pi$; those of θ, ϕ and $\pi - \phi$; and of ψ, 0 and 2π. Hence, doubling, since R may lie on either side of PQ, we have for the required chance

$$p = \frac{2}{\Delta^3}\int_0^r \int_{\theta=\phi}^{\theta=\pi-\phi} \int_0^{2\pi} \frac{\Delta}{r^2}(r - x \operatorname{cosec}\theta)^2 .\ 4x\,dx\,dt\,d\psi$$

$$= \frac{16\pi}{r^2\Delta^2}\int_0^r \int_{\theta=\phi}^{\theta=\pi-\phi} (r - x \operatorname{cosec}\theta)^3\, x\,dx\,dt$$

$$= \frac{32\pi}{r^2\Delta^2}\int_0^r \int_{\phi}^{\pi-\phi} (r - x \operatorname{cosec}\theta)^2 (1 - \theta\cot\theta)\, x^3\,dx \operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{2\pi r^4}{3\Delta^2}\int_0^{\frac{1}{2}\pi} \Big\{2(\pi - 2\phi)\sin 2\phi + 8\sin^2 2\phi$$
$$- 3\sin^2 2\phi \cos 2\phi + 64 \sin^4\phi \cos\phi \log\tan\tfrac{1}{2}\phi\Big\}\,d\phi$$

$$= \frac{2\pi^2 r^4}{5\Delta^2} = \frac{8\pi^2 r^2}{5s^2}.$$

When the polygon becomes a circle, $s = 2\pi r$, and we have $p = \frac{2}{5}$.

[See *Reprint*, Vol. VIII., p. 90; XIII., 17, 95; XVI., 50.]

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